Complementary Ramsey numbers, graph factorizations and Ramsey graphs

Akihiro Munemasa

Tohoku University

joint work with Masashi Shinohara

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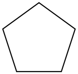
Ramsey Numbers

For a graph G,

$$\alpha(G) = \text{independence number} = \max\{\#\text{independent set}\}$$
$$\omega(G) = \text{clique number} = \max\{\#\text{clique}\} = \alpha(\bar{G}).$$



$$\omega(C_5) = \alpha(C_5) = 2.$$

$$\forall G \text{ with } 6 \text{ vertices, } \omega(G) \geq 3 \text{ or } \alpha(G) \geq 3.$$

These facts can be conveniently described by the Ramsey number:

$$R(3,3) = 6.$$

The smallest number of vertices required to guarantee $\alpha \geq 3$ or $\omega \geq 3$ (precise definition in the next slide).

Ramsey Numbers and a Generalization

The Ramsey number $R(m_1, m_2)$ is defined as:

$$
\begin{aligned}
&R(m_1, m_2) \\
&= \min\{n \mid |V(G)| = n \implies \omega(G) \geq m_1 \text{ or } \alpha(G) \geq m_2\} \\
&= \min\{n \mid |V(G)| = n \implies \omega(G) \geq m_1 \text{ or } \omega(\bar{G}) \geq m_2\} \\
&= \min\{n \mid |V(G)| = n \implies \alpha(\bar{G}) \geq m_1 \text{ or } \alpha(G) \geq m_2\}
\end{aligned}
$$

A graph with n vertices defines a partition of $E(K_n)$ into $\mathbf{2}$ parts, “edges” and “non-edges”.

Generalized Ramsey numbers $R(m_1, m_2, \ldots, m_k)$ can be defined if we consider partitions of $E(K_n)$ into $\boldsymbol{k}$ parts, i.e., edge-colorings.

Let $[n] = \{1, 2, \ldots, n\}$, and $E(K_n) = \binom{[n]}{2}$. The set of k-edge-coloring of K_n is denoted by $C(n, k)$:

$$C(n, k) = \{f \mid f : E(K_n) \to [k]\}.$$

We abbreviate

$$\omega_i(f) = \omega([n], f^{-1}(i)), \quad \alpha_i(f) = \alpha([n], f^{-1}(i)).$$

$$\begin{aligned} & R(m_1, \ldots, m_k) \\ & = \min\{n \mid \forall f \in C(n, k), \exists i \in [k], \omega_i(f) \geq m_i\} \\ & \bar{R}(m_1, \ldots, m_k) \\ & = \min\{n \mid \forall f \in C(n, k), \exists i \in [k], \alpha_i(f) \geq m_i\} \end{aligned}$$

The last one is called the complemtary Ramsey number.

$$\bar{R}(m_1, m_2) = R(m_2, m_1) = R(m_1, m_2).$$

So we focus on the case $k \geq 3$. Also we assume $m_i \geq 3$.

History

- We submitted to our work to a conference proceedings, and received positive reviews, in 2013.
- We uploaded our paper arXiv:1406.2050.
- David Conlon notified to us, that the concept was introduced already by Erdős–Hajnal–Rado (1965), some results were proved by Erdős–Szemerédi (1972).
- Chung–Liu (1978), "d-chromatic Ramsey numbers",
  $\bar{R}(m_1, \dots, m_k) = R^k_{k-1}(K_{m_1}, \dots, K_{m_k})$.
  $\bar{R}(4, 4, 4) = 10$.
- Harborth–Moller (1999), "weakened Ramsey numbers",
  $\bar{R}(m, \dots, m) = R^k_{k-1}(K_m)$.
- Xu–Shao–Su–Li (2009), "multigraph Ramsey numbers",
  $\bar{R}(m_1, \dots, m_k) = f^{(k-1)}(m_1, \dots, m_k)$. $\bar{R}(5, 5, 5) \geq 20$.

Geometric Application

Given a metric space (X, d) and a positive integer k, classify subsets Y of X with the largest size subject to

$$|\{d(x,y) \mid x,y \in Y,\ x \neq y\}| \leq k.$$

For example, $X = \mathbb{R}^n$, $k = 1 \implies$ regular simplex.
The method is by induction on k.

The distance function d defines a k-edge-coloring of the complete graph on Y.

If

$$\bar{R}(\underbrace{m, m, \dots, m}_{k}) \leq |Y|,$$

then Y must contain an m-subset having only $(k-1)$ distances (so we can expect to use already obtained results for $k-1$).

Einhorn and Schoenberg (1966)

Conjecture

The vertices of the regular icosahedron is the only 12-point 3-distance set in $\mathbb{R}^3$.

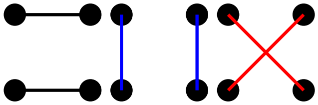
- Claimed to be proven by Shinohara, arXiv:1309.2047.
- It would simplify the proof if we had $\bar{R}(5,5,5) = 12$, but this was not the case.
- Xu–Shao–Su–Li (2009), $\bar{R}(5,5,5) \geq 20$.
- What is $\bar{R}(5,5,5)$? It should be easier than determining $\bar{R}(5,5) = R(5,5)$, which is known to satisfy

$$43 \leq R(5,5) \leq 48.$$

In fact,

$$R(m,m) = \bar{R}(m,m) \geq \bar{R}(m,m,m) \geq \bar{R}(m,m,m,m) \cdots .$$

$\bar{R}(3,3,3) = 5$ by factorization



- K_4 has a 3-edge-coloring f into $2K_2$ (a 1-factorization). Then $\alpha_i(f) = 2$ for $i = 1, 2, 3$. This implies

$$\bar{R}(3,3,3) > 4.$$

The argument can be generalized to give:

Theorem

If K_{mn} is factorable into k copies of nK_m, then $\bar{R}(\underbrace{n+1, \dots, n+1}_{k}) = mn + 1$.

Setting $m = n = 2$ and $k = 3$, we obtain $\bar{R}(3,3,3) = 5$.

Factorizations

Theorem

If K_{mn} is factorable into k copies of nK_m, then
$\bar{R}(\underbrace{n+1,\dots,n+1}_{k}) = mn+1$.

- Setting $m=3$, $n=2t+1$, $k=3t+1$, the existence of a Kirkman triple system in K_{3n} implies
$\bar{R}(\underbrace{2t+2,\dots,2t+2}_{3t+1}) = 6t+4$.
- Harborth–Möller (1999): Setting $m=n$, $k=n+1$, if $n-1$ MOLS of order n exist, then
$\bar{R}(\underbrace{n+1,\dots,n+1}_{n+1}) = n^2+1$.

The converse of the last statement also holds.

Ramsey (s,t)-graph

A graph G is said to be a Ramsey (s,t)-graph if

$$\omega(G) < s \text{ and } \alpha(G) < t.$$

We write $G \subset H$ if H is an edge-subgraph of G, and write $(V(G), E(G) \setminus E(H)) = G - H$.

Theorem

For $m_1, m_2, m_3, n \geq 2$, the following are equivalent.

(i) $\bar{R}(m_1, m_2, m_3) \leq n$,

(ii) for any two Ramsey (m_1, m_2)-graphs G and H on the vertex set $[n]$ such that $G \supset H$, one has $\alpha(G - H) \geq m_3$.

Small complementary Ramsey numbers

Chung–Liu (1978):

k	3	4	5	6	7	8
$\bar{R}(k,3,3)$	5	5	5	6	$\cdots$	$\cdots$
$\bar{R}(k,4,3)$	5	7	8	8	9	$\cdots$
$\bar{R}(k,5,3)$	5	8	9	11	12	12

and

$$\bar{R}(k,5,3) = \begin{cases} 13 & \text{if } 9 \le k \le 13, \\ 14 & \text{if } k \ge 14. \end{cases}$$

Small complementary Ramsey numbers

We abbreviate

$$\bar{R}(m;k) = \bar{R}(\underbrace{m,\dots,m}_{k}).$$

k	3	4	5	6	7	8	9	10	11 $\cdots$ 15	16
$\bar{R}(3;k)$	5	3	$\cdots$							
$\bar{R}(4;k)$	10	10	7	5	4	$\cdots$				
$\bar{R}(5;k)$	?	?	17	10	9	6	6	6	5 $\cdots$	
$\bar{R}(6;k)$	?	?	?	26	16	11	11	8	7 $\cdots$ 7	6

Thank you very much for your attention!