Imprimitive permutation groups which are nearly multiplicity-free

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Permutation groups

Let G be a finite group acting transitively on a finite set X.

$$G \text{ acts on } X \times X = R_0 \cup R_1 \cup \cdots \cup R_d \quad \text{(orbitals)},$$
$$\text{adjacency matrices} \qquad A_0, \; A_1, \; \ldots, \; A_d.$$

Then

$$A_0 = I \quad \text{(WLOG we may assume)},$$
$$\sum_{i=0}^{d} A_i = J \quad \text{(all-one matrix)},$$
$$\forall i, \; \exists i', \; A_i^\top = A_{i'},$$
$$\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle \text{ is closed under multiplication.}$$

Indeed, $\mathcal{A} = \mathrm{End}_G(\mathbb{C}^X)$.

Association schemes

If X is a finite set,

$$X \times X = R_0 \cup R_1 \cup \cdots \cup R_d \quad \text{(partition)},$$

adjacency matrices $\quad A_0, \; A_1, \; \ldots, \; A_d$

satisfy

$$A_0 = I \quad \text{(WLOG we may assume)},$$

$$\sum_{i=0}^{d} A_i = J \quad \text{(all-one matrix)},$$

$$\forall i, \exists i', \; A_i^\top = A_{i'},$$

$$\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle \text{ is closed under multiplication},$$

then $(X, \{R_i\}_{i=0}^{d})$ is called an association scheme, $\mathcal{A}$ is called its Bose-Mesner algebra.

Commutative Association schemes

An association scheme $(X, \{R_i\}_{i=0}^d)$ is commutative if its Bose-Mesner algebra $\mathcal{A}$ is commutative. If $(X, \{R_i\}_{i=0}^d)$ comes from a transitive permutation group G on X, then the permutation representation of G on $\mathbb{C}^X$ decomposes:

$$\begin{aligned}\mathbb{C}^X &= \bigoplus_{k=0}^{e} V_k \quad \text{(isotypic components)} \\ &= \bigoplus_{k=0}^{e} \bigoplus_{j=1}^{\mu_k} V_{kj} \quad \text{(irreducibles).}\end{aligned}$$

$$\mathcal{A} = \mathrm{End}_G(\mathbb{C}^X) = \bigoplus_{k=0}^{e} \mathrm{End}_G(V_k) \cong \bigoplus_{k=0}^{e} M_{\mu_k}(\mathbb{C})$$

which is commutative if and only if $\mu_k = 1$ for all k (multiplicity-free).

Bases of $\mathrm{End}_G(\mathbb{C}^X)$

$$\mathcal{A} = \mathrm{End}_G(\mathbb{C}^X) = \bigoplus_{k=0}^{e} \mathrm{End}_G(V_k) \cong \bigoplus_{k=0}^{e} M_{\mu_k}(\mathbb{C})$$

has two bases:

$$\{A_0, A_1, \ldots, A_d\}, \quad \bigcup_{k=0}^{e} \{E_k^{(i,j)} \mid 1 \le i, j \le \mu_k\}$$

(adjacency matrices) (matrix units of each component)

So

$$d + 1 = \sum_{k=0}^{e} \mu_k^2.$$

How do we find $E_k^{(i,j)}$?

Suppose $\mu_k = 2$, for example

If

$$V_k = V_{k,1} \oplus V_{k,2},$$

then

$$\mathrm{End}_G(V_k) = \begin{bmatrix} \mathrm{End}_G(V_{k,1}) & \mathrm{Hom}_G(V_{k,1}, V_{k,2}) \\ \mathrm{Hom}_G(V_{k,2}, V_{k,1}) & \mathrm{End}_G(V_{k,2}) \end{bmatrix},$$

$$\mathrm{End}_G(V_{k,j}) = \mathbb{C}\,\mathrm{id}_{V_{k,j}},$$

$$\dim \mathrm{Hom}_G(V_{k,1}, V_{k,2}) = \dim \mathrm{Hom}_G(V_{k,2}, V_{k,1}) = 1.$$

Basis of each of the four 1-dimensional spaces (unique up to scalar) $\to$ matrix units $E_k^{(i,j)}$.
Orthogonal decomposition of unitary representation $\to$ isometry in $\mathrm{Hom}_G(V_{k,1}, V_{k,2})$ is unique up to a complex number of modulus 1.
But the whole process depends on the decomposition of V_k.

If

$$V_k = V_{k,1} \quad \text{(irreducible)},$$

then

$$\mathrm{End}_G(V_k) = \mathrm{End}_G(V_{k,1}) = \mathbb{C}\,\mathrm{id}_{V_{k,1}}\,.$$

If multiplicity-free ($\mu_k = 1$ for all k), then

$$\mathrm{End}_G(\mathbb{C}^X) \cong \mathbb{C}\,\mathrm{id}_{V_{0,1}} \oplus \cdots \oplus \mathbb{C}\,\mathrm{id}_{V_{e,1}}$$

has a canonical basis

$$\{E_0, \dots, E_e\} \cong \{\mathrm{id}_{V_{0,1}}, \dots, \mathrm{id}_{V_{e,1}}\}$$

(so $e = d$), where E_k is the orthogonal projection onto $V_k = V_{k,1}$.
The first eigenmatrix $P = (P_{ij})$ is defined by

$$A_j = \sum_{i=0}^{d} P_{ij} E_i.$$

Nearly multiplicity-free?

$$\mathbb{C}^X = \bigoplus_{k=0}^{e} V_k, \quad V_k = \bigoplus_{j=1}^{\mu_k} V_{k,j}.$$

Suppose

$$\mu_k \le 2$$

and W is a G-submodule of $\mathbb{C}^X$ such that

W contains an isomorphic copy of $V_{k,1}$
exactly once whenever $\mu_k = 2$.

Then we can define that copy to be $V_{k,1}$, and decompose V_k as

$$V_k = V_{k,1} \oplus V_{k,1}^{\perp} \quad (\perp \text{ inside } V_k)$$

Nearly multiplicity-free imprimitive perm. group

$$\mathbb{C}^{X} = \bigoplus_{k=0}^{e} V_k, \quad V_k = \bigoplus_{j=1}^{\mu_k} V_{k,j}.$$

Suppose

$$\mu_k \le 2$$

and W is the G-submodule of $\mathbb{C}^X$ consisting of functions on X which are constant on blocks. Suppose

W contains an isomorphic copy of $V_{k,1}$
exactly once whenever $\mu_k = 2$.

Then we can decompose V_k canonically, subject to the choice of a system of imprimitivity (blocks).

Group theoretically...

Let

$$G \geq H \geq K$$

be finite groups. Then the permutation character 1_K^G is the sum

$$1_K^G = 1_H^G + (1_K^G - 1_H^G),$$

so it makes sense to consider the situation where

1_H^G and $(1_K^G - 1_H^G)$ are both multiplicity-free.

This means that, if χ is an irreducible character of G appearing in 1_K^G with multiplicity $(1_K^G, \chi) > 1$, then

$$(1_H^G, \chi) = (1_K^G - 1_H^G, \chi) = 1.$$

$G \geq H \geq K$

G acts on $X = G/K$, and H defines a G-invariant equivalence relation $\sim$ of X. Suppose

$$1_K^G = \cdots + 2\chi_k + \cdots$$
$$1_H^G = \cdots + \chi_k + \cdots .$$

Then

$$\mathbb{C}^X = \mathbb{C}^{G/K} = \cdots \oplus V_k \oplus \cdots$$
$$\mathbb{C}^{X/\sim} = \mathbb{C}^{G/H} = \cdots \oplus V_{k,1} \oplus \cdots$$

so

$$V_k = V_{k,1} \oplus V_{k,1}^{\perp} \quad (\perp \text{ inside } V_k),$$

$V_{k,1}$ and $V_{k,1}^{\perp}$ are isomorphic irreducibles.

An (imprimitive) permutation group G is nearly multiplicity-free if both 1_H^G and $1_K^G - 1_H^G$ are multiplicity free, where

$$K = \text{point stabilizer}, \quad H = \text{block stabilizer}.$$

Write

$$1_K^G = \sum_{k=0}^{e} \mu_k \chi_k, \quad \mu_k \in \{1, 2\},$$

$$\mathbb{C}^{G/K} = \bigoplus_{k=0}^{e} V_k \quad \text{(isotypic components)}$$

$$\supset W = \mathbb{C}^{G/H}.$$

Then $\mathrm{End}_G(\mathbb{C}^{G/K})$ has a basis

$$\{\mathrm{id}_{V_k} \mid \mu_k = 1\} \cup \{\mathrm{id}_{V_k \cap W}, \mathrm{id}_{V_k \cap W^\perp}, \beta_k, \beta_k^* \mid \mu_k = 2\},$$

where $\mathrm{Hom}_G(V_k \cap W, V_k \cap W^\perp) = \mathbb{C}\beta_k$,
$\mathrm{Hom}_G(V_k \cap W^\perp, V_k \cap W) = \mathbb{C}\beta_k^*$,

First eigenmatrix

The square matrix P tabulating the coefficients of the obtained basis in A_j:

$$
\begin{array}{r}
\\
\mathrm{id}_{V_k} \text{ with } \mu_k = 1 \\
\\
\mathrm{id}_{V_k \cap W} \\
\beta_k \\
\beta_k^* \\
\mathrm{id}_{V_k \cap W^\perp}
\end{array}
\begin{array}{c}
A_j \\
\left[\begin{array}{c}
P_{kj} \\
\\
\hline
P_{kj}^{(1,1)} \\
P_{kj}^{(1,2)} \\
P_{kj}^{(2,1)} \\
P_{kj}^{(2,2)}
\end{array}\right]
\end{array}
= P
$$

$$
A_j = \sum_{\mu_k=1} P_{kj} E_k + \sum_{\mu_k=2} \sum_{i_1,i_2 \in \{1,2\}} P_{kj}^{(i_1,i_2)} E_k^{(i_1,i_2)},
$$

where $E_k \leftrightarrow \mathrm{id}_{V_k}$, $E_k^{(1,1)} \leftrightarrow \mathrm{id}_{V_k \cap W}$, $E_k^{(2,2)} \leftrightarrow \mathrm{id}_{V_k \cap W^\perp}$, $E_k^{(2,1)} \leftrightarrow \beta_k$, $E_k^{(1,2)} \leftrightarrow \beta_k^*$.

Second eigenmatrix $Q = |X|P^{-1}$

Using the entries of Q, one can express E_k, $E_k^{(i,j)}$ as a linear combination of A_j's.
Let $\phi_Y \in \mathbb{C}^X$ be the characteristic vector of a subset $Y \subset X$.
Multiplicity-free case: Delsarte's inequalities

$$0 \le \|E_k \phi_Y\|^2 = \phi_Y^* E_k \phi_Y = \sum_{j=0}^{d} Q_{jk} \frac{1}{|X|} \phi_Y^* A_j \phi_Y$$

Nearly multiplicity-free case: take

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \ge 0 \quad (\overline{a_{12}} = a_{21}),$$

and set

$$E = \sum_{i,j} a_{ij} E_k^{(i,j)}.$$

Nearly multiplicity-free case: take

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \geq 0 \quad (\overline{a_{12}} = a_{21}),$$

and set

$$E = \sum_{i_1,i_2} a_{i_1,i_2} E_k^{(i_1,i_2)}.$$

Since

$$E_k^{(i_1,i_2)} = \frac{1}{|X|} \sum_{j=0}^{d} Q_{j,(k,i_1,i_2)} A_j,$$

we have

$$0 \leq \phi_Y^* E \phi_Y = \sum_{i_1,i_2} a_{i_1,i_2} \phi_Y^* E_k^{(i_1,i_2)} \phi_Y$$

$$= \sum_{i_1,i_2} a_{i_1,i_2} \sum_{j=0}^{d} Q_{j,(k,i_1,i_2)} \frac{1}{|X|} \phi_Y^* A_j \phi_Y.$$

Bannai-Ito, Section II.11

If

$$\mathbb{C}^X = \bigoplus_{k=0}^{e} V_k \quad \text{(isotypic components)}$$

and E_k is the projection of $\mathbb{C}^X$ onto V_k, then

$$E_k = \frac{1}{|X|} \sum_{j=0}^{d} q_k(j) A_j,$$

$$q_k(j) = \frac{|X|\chi_k(1)}{|G|k_j} \sum_{g \in K\alpha_j K} \overline{\chi_k(g)}.$$

This holds without assuming multiplicity-freeness.
Nearly multiplicity-free case: if $V_k = V_{k,1} \oplus V_{k,2}$, then

$$E_k = \mathrm{id}_{V_k} = \mathrm{id}_{V_{k,1}} + \mathrm{id}_{V_{k,2}} = E_k^{(1,1)} + E_k^{(2,2)}.$$

Recall $W \subset \mathbb{C}^X$ is the submodule affording 1_H^G.

Bannai-Ito, Section II.11

$$E_k = \frac{1}{|X|}\sum_{j=0}^{d} q_k(j) A_j, \quad q_k(j) = \frac{|X|\chi_k(1)}{|G|k_j}\sum_{g\in K\alpha_j K}\overline{\chi_k(g)}.$$

Nearly multiplicity-free case: if $V_k = V_{k,1} \oplus V_{k,2}$, then

$$E_k = \mathrm{id}_{V_k} = \mathrm{id}_{V_{k,1}} + \mathrm{id}_{V_{k,2}} = E_k^{(1,1)} + E_k^{(2,2)}.$$

Recall $W \subset \mathbb{C}^X$ is the submodule affording 1_H^G.

$$\begin{aligned} E_k^{(1,1)} &= \text{projection onto } V_{k,1} = V_k \cap W \\ &= E_k \cdot (\text{projection onto } W) \\ &= E_k \cdot (\text{averaging over blocks}) \end{aligned}$$

Then $E_k^{(2,2)} = E_k - E_k^{(1,1)}$.

Matrix units $E_k^{(1,1)}, E_k^{(1,2)}, E_k^{(2,1)}, E_k^{(2,2)}$

1. $E_k = E_k^{(1,1)} + E_k^{(2,2)}$ can be computed using group characters.
2. $E_k^{(1,1)} = E_k \cdot$ (averaging over blocks)
3. $E_k^{(2,2)} = E_k - E_k^{(1,1)}$

$E_k^{(1,2)}$ should be found by normalizing

$$E_k^{(1,1)} A_j E_k^{(2,2)}.$$

This can be regarded as an element of $\mathrm{Hom}_G(V_{k,2}, V_{k,1})$, and $\exists j$ such that $E_k^{(1,1)} A_j E_k^{(2,2)} \neq 0$,
since $E_k^{(1,1)} \mathrm{End}_G(\mathbb{C}^X) E_k^{(2,2)} \neq 0$.

$S_n \geq S_{n-2} \times S_2 \geq S_{n-2}$

G/H = Johnson scheme $J(n, 2)$,
G/K = ordered pairs $X = \{(i, j) \mid 1 \leq i, j \leq n,\ i \neq j\}$.

$$
\begin{aligned}
1_H^G &= \chi_n + \chi_{n-1,1} + \chi_{n-2,2}, \\
1_K^G &= \chi_n + 2\chi_{n-1,1} + \chi_{n-2,2} + \chi_{n-2,1,1}, \\
\mathbb{C}^X &= V_0 \oplus (V_{1,1} \oplus V_{1,2}) \oplus V_2 \oplus V_3, \\
J(n,2):&\ V_0 \oplus V_{1,1} \oplus V_2 \subset \mathbb{C}^X.
\end{aligned}
$$

Two bases:

$$
\begin{gathered}
A_0, A_1, A_2, A_3, A_4, A_5, A_6, \\
E_0, E_1^{(1,1)}, E_1^{(1,2)}, E_1^{(2,1)}, E_1^{(2,2)}, E_2, E_3.
\end{gathered}
$$

First eigenmatrix P

The jth column of the matrix P consists of the coefficients of A_j when written as a linear combination of E's.

$$\begin{array}{r} E_0 \\ E_1^{(1,1)} \\ E_1^{(1,2)} \\ E_1^{(2,1)} \\ E_1^{(2,2)} \\ E_2 \\ E_3 \end{array} \begin{bmatrix} 1 & 1 & n-2 & n-2 & n-2 & n-2 & (n-2)(n-3) \\ 1 & 1 & \frac{n-4}{2} & \frac{n-4}{2} & \frac{n-4}{2} & \frac{n-4}{2} & -2(n-3) \\ 0 & 0 & m & -m & -m & m & 0 \\ 0 & 0 & m & m & -m & -m & 0 \\ 1 & -1 & \frac{n-2}{2} & -\frac{n-2}{2} & \frac{n-2}{2} & -\frac{n-2}{2} & 0 \\ 1 & 1 & -1 & -1 & -1 & -1 & 2 \\ 1 & -1 & -1 & 1 & -1 & 1 & 0 \end{bmatrix}$$

where

$$m = \frac{\sqrt{n(n-2)}}{2}.$$

Remark

System of imprimitivity may not be unique:

$$S_n \geq \begin{cases} S_2 \times S_{n-2} \\ S_{n-1} \end{cases} \geq S_{n-2}.$$

One could define "multiplicity-free chain"

$$G \geq H_1 \geq H_2 \geq \cdots$$

$$1_{H_1}^G, 1_{H_2}^G - 1_{H_1}^G, 1_{H_3}^G - 1_{H_2}^G, \ldots \text{ are multiplicity-free}$$

However, a different definition is known (in a book by Ceccherini-Silberstein, Scarabotti and Tolli):

$$1_{H_1}^G, 1_{H_2}^{H_1}, 1_{H_3}^{H_2}, \ldots \text{ are multiplicity-free}$$