Turyn's construction of conference matrices

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An exercise

Notation

For a positive integer k, write

$$\{1, 2, \ldots, k\} = [k].$$

Exercise

Let A be a finite set, and let $\varphi : A \to \mathbb{Z}$ be a function. Let A_i $(i = 1, 2, \ldots, k)$ be subsets of A. Then

$$\sum_{a \in \bigcup_{j \in [k]} A_j} \varphi(a) = \sum_{\substack{J \subset [k] \\ J \neq \emptyset}} (-1)^{|J|-1} \sum_{a \in \bigcap_{j \in J} A_j} \varphi(a).$$

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For the special case $k = 2$,

$$\sum_{a\in\bigcup_{j\in[2]} A_j} \varphi(a) = \sum_{\substack{J\subset[2]\\ J\neq\emptyset}} (-1)^{|J|-1} \sum_{a\in\bigcap_{j\in J} A_j} \varphi(a).$$

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This means

$$\sum_{a\in A_1\cup A_2} \varphi(a) = \sum_{J\in\{\{1\},\{2\},\{1,2\}\}} (-1)^{|J|-1} \sum_{a\in\bigcap_{j\in J} A_j} \varphi(a).$$

$$= \sum_{a\in A_1} \varphi(a) + \sum_{a\in A_2} \varphi(a) - \sum_{a\in A_1\cap A_2} \varphi(a).$$

$\sum_{a\in\bigcup_{j\in[k]} A_j} \varphi(a) = \sum_{\substack{J\subset[k]\\ J\neq\emptyset}}(-1)^{|J|-1}\sum_{a\in\bigcap_{j\in J} A_j} \varphi(a)$

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$$\sum_{a\in\bigcup_{j\in[k]} A_j} 1 = \sum_{\substack{J\subset[k]\\ J\neq\emptyset}}(-1)^{|J|-1}\sum_{a\in\bigcap_{j\in J} A_j} 1.$$

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$$\left|\bigcup_{j\in[k]} A_j\right| = \sum_{\substack{J\subset[k]\\ J\neq\emptyset}}(-1)^{|J|-1}\left|\bigcap_{j\in J} A_j\right|.$$

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The inclusion-exclusion principle.

An exercise

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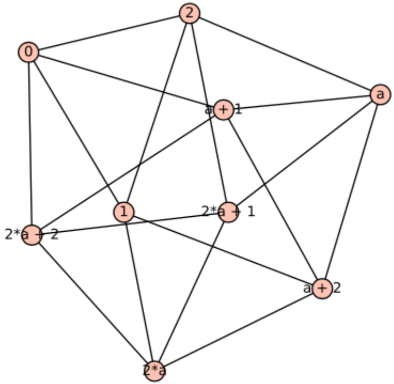
As in the inclusion-exclusion principle, give a proof by induction.

Conference graphs



9 vertices, $\frac{9-1}{2} = 4$ neighbors

$$\#\text{ common neighbors} = \begin{cases} \frac{4}{2} = 2 & \text{if non-adjacent,} \\ \frac{4}{2} - 1 = 1 & \text{if adjacent.} \end{cases}$$

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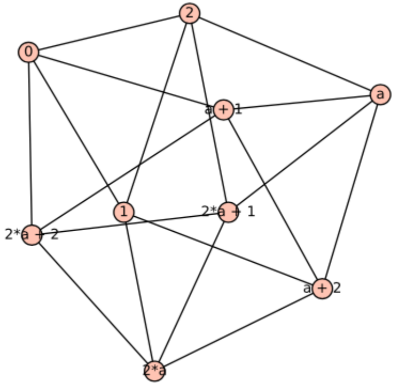

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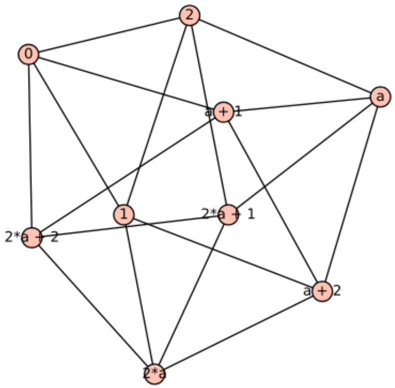

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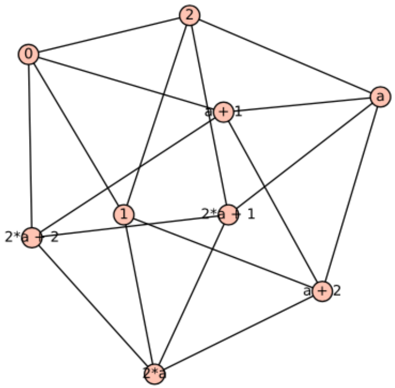

n vertices, $\frac{n-1}{2}$ neighbors

$$\# \text{ common neighbors} = \begin{cases} \frac{\frac{n-1}{2}}{2} = \frac{n-1}{4} & \text{if non-adjacent,} \\ \frac{\frac{n-1}{2}}{2} - 1 = \frac{n-5}{4} & \text{if adjacent.} \end{cases}$$

Conference graphs

Definition

A graph on n vertices is called a conference graph if

- every vertex has $(n-1)/2$ neighbors,
- every pair of non-adjacent vertices has $(n-1)/4$ common neighbors,
- every pair of adjacent vertices has $(n-5)/4$ common neighbors.

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In terms of adjacency matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{otherwise} \end{cases}$$

$$A^2 = \frac{n-1}{2} I + \frac{n-5}{4} A + \frac{n-1}{4} (J - I - A),$$

where J is the all-one matrix.

Conference matrices

Let A be a $(0,1)$-matrix satisfying

$$A^2 = \frac{n-1}{2}I + \frac{n-5}{4}A + \frac{n-1}{4}(J - I - A).$$

Set

$$W = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & 2A - J + I \end{bmatrix}$$

Then W is a $(n+1) \times (n+1)$ matrix all of whose entries are $0, \pm 1$, and

$$W = W^\top, \quad W^2 = WW^\top = nI.$$

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Exercise

Verify $W^2 = nI$.

Conference matrices

Definition

A symmetric $(0, \pm 1)$-matrix W of order $n+1$ all of whose diagonal entries are 0 is called a conference matrix if $W^2 = nI$.

If W is a conference matrix, then there exists a diagonal matrix D all of whose diagonal entries are ± 1 such that

$$DWD = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & C \end{bmatrix}$$

for some $(0, \pm 1)$-matrix C.

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Exercise

Show that C is of the form $2A - J + I$ for the adjacency matrix A of some conference graph.

Existence of conference matrices

Note that $(n-1)/4$ was the number of common neighbors of two non-adjacent vertices in a conference graph on n vertices.

Problem

Does there exists a conference matrix of order $n+1$ whenever n is a positive integer with $n \equiv 1 \pmod 4$?

The smallest n for which the answer is unknown is $n = 65$. The answer is "yes" if n is a prime power.

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This is also the smallest number of vertices for which the existence is undecided for a parameters (k, λ, μ) for a strongly regular graph.

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Goldberg (1966)
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Define

$$D = C \otimes C \otimes C - I \otimes J \otimes C - C \otimes I \otimes J - J \otimes C \otimes I$$

Exercise

$D^2 = n^3 I - J$ and hence $\tilde{W} = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & D \end{bmatrix}$: conference matrix.

$C(n+1) \neq \emptyset \implies C(n^3+1) \neq \emptyset$

$W = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & C \end{bmatrix}$: $(n+1) \times (n+1)$ matrix with entries in $\{0, \pm 1\}$,

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Then

$$D = \begin{matrix} C \otimes C \otimes C \\ -I \otimes J \otimes C \\ -C \otimes I \otimes J \\ -J \otimes C \otimes I \end{matrix} \ \text{satisfies} \ \begin{matrix} \text{disjoint} \implies (0, \pm 1) \text{ matrix} \\ D \circ I = 0, \\ DJ = 0, \\ D^2 = n^3 I - J. \end{matrix}$$

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Assume C: $n \times n$ matrix with entries in $\{0, \pm 1\}$.

$$C = C^\top, \quad CJ = 0, \quad C \circ I = 0, \quad C^2 = nI - J$$

$D = B_0 + B_1 + B_2 + B_3$, where

$$\begin{aligned} B_0 &= C \otimes C \otimes C \\ B_1 &= -I \otimes J \otimes C \\ B_2 &= -C \otimes I \otimes J \\ B_3 &= -J \otimes C \otimes I \end{aligned}$$

$$D^2 = (B_0 + B_1 + B_2 + B_3)^2 = n^3 I - J.$$

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$D = B_0 + B_1 + B_2 + B_3$, where

$$\begin{aligned} B_0^2 &= (nI - J) \otimes (nI - J) \otimes (nI - J) \\ B_1^2 &= I \otimes nJ \otimes (nI - J) \\ B_2^2 &= (nI - J) \otimes I \otimes nJ \\ B_3^2 &= nJ \otimes (nI - J) \otimes I \end{aligned}$$

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$$C = C^\top, \quad CJ = 0, \quad C \circ I = 0, \quad C^2 = nI - J$$

$D = B_0 + B_1 + B_2 + B_3$, where

$$\begin{aligned}
B_0^2 &= (nI - J) \otimes (nI - J) \otimes (nI - J) \\
B_1^2 &= -nI \otimes (-J) \otimes (nI - J) \\
B_2^2 &= -(nI - J) \otimes nI \otimes (-J) \\
B_3^2 &= -(-J) \otimes (nI - J) \otimes nI
\end{aligned}$$

$$\begin{aligned}
D^2 &= (B_0 + B_1 + B_2 + B_3)^2 \\
&= B_0^2 + B_1^2 + B_2^2 + B_3^2
\end{aligned}$$

$$
\begin{aligned}
B_0^2 &= (x_1 + y_1)(x_2 + y_2)(x_3 + y_3) \\
B_1^2 &= -x_1 y_2 (x_3 + y_3) \\
B_2^2 &= -(x_1 + y_1) x_2 y_3 \\
B_3^2 &= -y_1 (x_2 + y_2) x_3
\end{aligned}
$$

sum $= x_1 x_2 x_3 + y_1 y_2 y_3$

$$
\begin{aligned}
B_0^2 &= (x_1 + y_1)(x_2 + y_2)(x_3 + y_3) \\
B_1^2 &= -x_1 y_2 (x_3 + y_3) \\
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B_3^2 &= -y_1 (x_2 + y_2) x_3
\end{aligned}
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sum $= x_1 x_2 x_3 + y_1 y_2 y_3$

$$
\begin{aligned}
D^2 &= B_0^2 + B_1^2 + B_2^2 + B_3^2 \\
&= nI \otimes nI \otimes nI + (-J) \otimes (-J) \otimes (-J). \\
&= n^3 I - J.
\end{aligned}
$$

$$\begin{aligned} B_0^2 &= (x_1+y_1)(x_2+y_2)(x_3+y_3) \\ B_1^2 &= -x_1y_2(x_3+y_3) \\ B_2^2 &= -(x_1+y_1)x_2y_3 \\ B_3^2 &= -y_1(x_2+y_2)x_3 \end{aligned}$$

sum $= x_1x_2x_3 + y_1y_2y_3$

$$\begin{aligned} D^2 &= B_0^2 + B_1^2 + B_2^2 + B_3^2 \\ &= nI \otimes nI \otimes nI + (-J) \otimes (-J) \otimes (-J). \\ &= n^3 I - J. \end{aligned}$$

Exercise

$D^2 = n^3 I - J$ and hence $\tilde{W} = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & D \end{bmatrix}$: conference matrix.

$$
\begin{aligned}
B_0^2 &= (x_1+y_1)(x_2+y_2)(x_3+y_3) \\
B_1^2 &= -x_1 y_2 (x_3+y_3) \\
B_2^2 &= -(x_1+y_1) x_2 y_3 \\
B_3^2 &= -y_1 (x_2+y_2) x_3
\end{aligned}
$$

sum $= x_1x_2x_3 + y_1y_2y_3$

$$
\begin{aligned}
D^2 &= B_0^2 + B_1^2 + B_2^2 + B_3^2 \\
&= nI \otimes nI \otimes nI + (-J) \otimes (-J) \otimes (-J). \\
&= n^3 I - J.
\end{aligned}
$$

Seberry (1969) found analogous construction for $C(n^5+1), C(n^7+1)$.
Different method by Belevitch (1950) for $C(n^2+1)$.

Turyn (1971)

Theorem

$$C(n+1) \neq \emptyset \implies C(n^k+1) \neq \emptyset$$

for any odd positive integer k.

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$$D = C \otimes \cdots \otimes C$$

$$\ldots$$

Turyn (1971)

Theorem

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for any odd positive integer k.

$k = 3$:

$$D = \begin{matrix} C \otimes C \otimes C \\ -I \otimes J \otimes C \\ -C \otimes I \otimes J \\ -J \otimes C \otimes I \end{matrix} \quad \text{satisfies} \quad \begin{matrix} DJ = D \circ I = 0, \\ D^2 = n^3 I - J. \end{matrix}$$

$$\implies \tilde{W} = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & D \end{bmatrix} : \text{ symmetric conference matrix.}$$

Turyn (1971)

Theorem

$$C(n+1) \neq \emptyset \implies C(n^k+1) \neq \emptyset$$

for any odd positive integer k.

$$D = C \otimes \cdots \otimes C$$
$$- \sum \text{replace some } C \otimes C \text{ with } I \otimes J$$

Turyn (1971)

Theorem

$$C(n+1) \neq \emptyset \implies C(n^k+1) \neq \emptyset$$

for any odd positive integer k.

$$D = C \otimes \cdots \otimes C$$

$$- \sum_{t=1}^{(k-1)/2} \text{replace } t\ C \otimes C\text{'s with } I \otimes J\text{'s}$$

Turyn (1971)

Theorem

$$C(n+1) \neq \emptyset \implies C(n^k+1) \neq \emptyset$$

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Summands are disjoint, orthogonal

Turyn (1971)

Theorem

$$C(n+1) \neq \emptyset \implies C(n^k+1) \neq \emptyset$$

for any odd positive integer k.

$$D = C \otimes \cdots \otimes C$$
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Summands are disjoint, orthogonal $D^2 = n^k I - J$?

$CJ = 0$, $C \circ I = 0$, $C^2 = nI - J$, $J^2 = nJ$

$$\begin{aligned} D = {} & C \otimes \cdots \otimes C \\ & - \sum \text{ replace } t \ C \otimes C\text{'s with } I \otimes J\text{'s} \end{aligned}$$

$$\begin{aligned} D^2 = {} & C^2 \otimes \cdots \otimes C^2 \\ & + \sum (\text{replace } t \ C \otimes C\text{'s with } I \otimes J\text{'s})^2 \end{aligned}$$

$CJ = 0$, $C \circ I = 0$, $C^2 = nI - J$, $J^2 = nJ$

$$D = C \otimes \cdots \otimes C$$
$$- \sum \text{replace } t \; C \otimes C\text{'s with } I \otimes J\text{'s}$$

$$D^2 = C^2 \otimes \cdots \otimes C^2$$
$$+ \sum (\text{replace } t \; C \otimes C\text{'s with } I \otimes J\text{'s})^2$$

$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J)$$
$$+ \sum_t \text{replace } t \; (nI - J) \otimes (nI - J)\text{'s with } I \otimes nJ\text{'s}$$

$CJ = 0$, $C \circ I = 0$, $C^2 = nI - J$, $J^2 = nJ$

$$D^2 = C^2 \otimes \cdots \otimes C^2$$
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$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J)$$
$$+ \sum_t \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } nI \otimes J\text{'s}$$

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$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J)$$
$$+ \sum_t \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } nI \otimes J\text{'s}$$

$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J)$$
$$+ \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } nI \otimes (-J)\text{'s}$$

$CJ = 0$, $C \circ I = 0$, $C^2 = nI - J$, $J^2 = nJ$

$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J) + \sum_t \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } I \otimes nJ\text{'s}$$

$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J) + \sum_t \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } nI \otimes J\text{'s}$$

$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J) + \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } nI \otimes (-J)\text{'s}$$

$x_i = nI$, $y_i = -J$

$$(x_1+y_1)\otimes\cdots\otimes(x_k+y_k)$$
$$+\sum_{t=1}^{(k-1)/2}(-1)^t \text{replace } t\ (x_i+y_i)\otimes(x_{i+1}+y_{i+1})\text{'s with } x_i\otimes y_{i+1}\text{'s}$$
$$= x_1\otimes\cdots\otimes x_k + y_1\otimes\cdots\otimes y_k$$

or equivalently,

$$(x_1+y_1)\cdots(x_k+y_k)-(x_1\cdots x_k+y_1\cdots y_k)$$
$$=-\sum_{t=1}^{(k-1)/2}(-1)^t \text{replace } t\ (x_i+y_i)(x_{i+1}+y_{i+1})\text{'s with } x_iy_{i+1}\text{'s}$$

$$(x_1 + y_1) \otimes \cdots \otimes (x_k + y_k)$$
$$+ \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t \ (x_i + y_i) \otimes (x_{i+1} + y_{i+1})\text{'s with } x_i \otimes y_{i+1}\text{'s}$$
$$= x_1 \otimes \cdots \otimes x_k + y_1 \otimes \cdots \otimes y_k$$

or equivalently,

$$(x_1 + y_1) \cdots (x_k + y_k) - (x_1 \cdots x_k + y_1 \cdots y_k)$$
$$= - \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t \ (x_i + y_i)(x_{i+1} + y_{i+1})\text{'s with } x_i y_{i+1}\text{'s}$$

This is a consequence of the inclusion-exclusion, or more generally, the Möbius inversion.

Inclusion-Exclusion

$\varphi : A \to M$, M: abelian group, $A_1, \dots, A_k \subset A$.

$$\sum_{a \in \bigcup_{i=1}^k A_i} \varphi(a) = \sum_{t=1}^{k} (-1)^{t-1} \sum_{|T|=t} \sum_{a \in \bigcap_{i \in T} A_i} \varphi(a).$$

$$(x_1 + y_1) \cdots (x_k + y_k) - (x_1 \cdots x_k + y_1 \cdots y_k)$$
$$= - \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t \ (x_i + y_i)(x_{i+1} + y_{i+1})\text{'s with } x_i y_{i+1}\text{'s}$$

Inclusion-Exclusion

$$\sum_{a\in\bigcup_{i=1}^k A_i} \varphi(a) = \sum_{t=1}^{k}(-1)^{t-1}\sum_{|T|=t}\sum_{a\in\bigcap_{i\in T} A_i} \varphi(a).$$

Let $A = \{0,1\}^k = (\bigcup_i A_i) \cup \{(0,\dots,0),(1,\dots,1)\}$, where

$$A_i = \{(a_1,\dots,a_k)\in A \mid (a_i,a_{i+1}) = (0,1)\},$$

$$\varphi(a_1,\dots,a_k) = \left(\begin{cases} x_1 & (a_1=0) \\ y_1 & (a_1=1)\end{cases}\right)\cdots\left(\begin{cases} x_k & (a_k=0) \\ y_k & (a_k=1)\end{cases}\right)$$

Then

$$(x_1+y_1)\cdots(x_k+y_k) - (x_1\cdots x_k + y_1\cdots y_k)$$

$$= -\sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t\ ({\color{red}x_i}+y_i)(x_{i+1}+{\color{red}y_{i+1}})\text{'s with } {\color{red}x_i y_{i+1}}\text{'s}$$

Thus

$$
\begin{aligned}
&(x_1 + y_1) \otimes \cdots \otimes (x_k + y_k) \\
&+ \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t \ (x_i + y_i) \otimes (x_{i+1} + y_{i+1})\text{'s with } x_i \otimes y_{i+1}\text{'s} \\
&= x_1 \otimes \cdots \otimes x_k + y_1 \otimes \cdots \otimes y_k
\end{aligned}
$$

This implies (by setting $x_i = nI$, $y_i = -J$)

$$D^2 = n^k I - J,$$

hence

$$C(n^k + 1) \neq \emptyset.$$

A weighing matrix of order n and weight w, is a $(0, \pm 1)$ matrix W satisfying $WW^\top = wI$.
Let $W(n, w)$ denote the set of weighing matrices of order n and weight w. Then

$$C(n+1) = W(n+1, n).$$

Theorem

Let k be odd.

$$W(n_i + 1, w) \neq \emptyset \ (i = 1, \dots, k) \implies W(n_1 n_2 \cdots n_k + 1, w^k) \neq \emptyset.$$

Originally formulated by Craigen (1992).

What if k is even? (Belevitch 1950, $k = 2$).