Oriented covers of the triangular graphs

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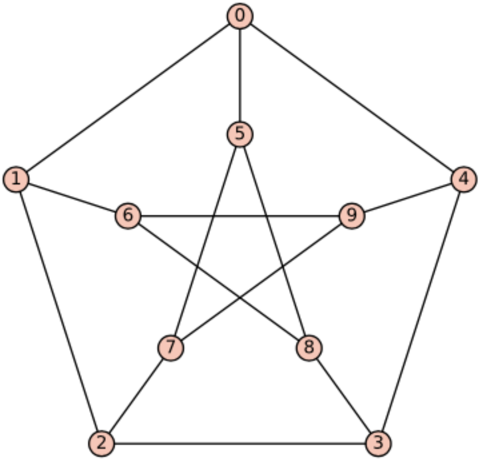
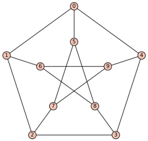
The Petersen graph



The Petersen graph



Definition:

$$V = \{\{i,j\} \mid i,j \in \{1,\dots,5\},\ i \neq j\},$$
$$E = \{\{\{i,j\},\{k,l\}\} \mid \{i,j\} \cap \{k,l\} = \emptyset\}.$$

Properties:

- 10 vertices,
- valency 3,
- no triangle or quadrangle
- diameter 2

Characterization: Properties $\Longrightarrow$ Unique.

Generalization $\longleftarrow$ next slide.

Triangular graph $T(n)$

Petersen graph:

$$V = \{\{i,j\} \mid i,j \in \{1,\dots,5\},\ i \neq j\},$$
$$E = \{\{\{i,j\},\{k,l\}\} \mid \{i,j\} \cap \{k,l\} = \emptyset\},$$
$$\overline{E} = \{\{\{i,j\},\{k,l\}\} \mid |\{i,j\} \cap \{k,l\}| = 1\}.$$

Triangular graph $T(n)$ $(n \geq 4)$:

$$V = \{\{i,j\} \mid i,j \in \{1,\dots,n\},\ i \neq j\},$$
$$E = \{\{\{i,j\},\{k,l\}\} \mid |\{i,j\} \cap \{k,l\}| = 1\}.$$

Properties:

- there are $n(n-1)/2$ vertices,
- valency is $2(n-2)$,
- each edge is contained in $\lambda = n - 2$ triangles,
- each pair of non-adjacent vertices has $\mu = 4$ common neighbors.

Characterization (Chang, 1959): Properties $\implies$ Unique unless $n = 8$.

Strongly regular graphs

Definition

A graph Γ is called a strongly regular graph (SRG) with parameters (k, λ, μ) if

- valency is k,
- each edge is contained in λ triangles,
- each pair of non-adjacent vertices has μ common neighbors.

$T(n)$ is a SRG with parameters $(2(n-2), n-2, 4)$.
The Petersen graph $\overline{T(5)}$ is a SRG with parameters $(3, 0, 1)$.
The complement of a SRG is again a SRG.

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Algebraic Graph Theory

For a regular graph Γ, the following are equivalent:

- Γ is strongly regular,
- the adjacency matrix has 3 distinct eigenvalues.

Example:

- The Petersen graph has spectrum $\{[3], [-2]^4, [1]^5\}$,
- $T(n)$ has spectrum $\{[2(n-2)], [n-4]^{n-1}, [-2]^{n(n-3)/2}\}$.
- $T(5)$ has spectrum $\{[6], [1]^4, [-2]^5\}$.

Simultaneous diagonalization

- The Petersen graph $\overline{T(5)}$ has spectrum $\{[3], [-2]^4, [1]^5\}$,
- $T(5)$ has spectrum $\{[6], [1]^4, [-2]^5\}$.

$$\begin{array}{ccccc} & & T(5) & & \overline{T(5)} \\ \color{red}{I} & + & \color{red}{A} & + & \color{red}{J-I-A} \\ \begin{pmatrix} [1] & & \\ & [1]^4 & \\ & & [1]^5 \end{pmatrix} & + & \begin{pmatrix} [6] & & \\ & [1]^4 & \\ & & [-2]^5 \end{pmatrix} & + & \begin{pmatrix} [3] & & \\ & [-2]^4 & \\ & & [1]^5 \end{pmatrix} \end{array}$$

Since A and J commute, they can be simultaneously diagonalized. The list of eigenvalues can be tabulated in a matrix form, and it is called the eigenmatrix:

$$\begin{pmatrix} 1 & 6 & 3 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

Symmetric association schemes

If X is a finite set,

$$X \times X = R_0 \cup R_1 \cup \cdots \cup R_d \quad \text{(partition)},$$

adjacency matrices $\quad A_0, \ A_1, \ \ldots, \ A_d$

satisfy

$$A_0 = I,$$

$$\sum_{i=0}^{d} A_i = J \quad \text{(all-one matrix)},$$

$$\forall i, \ A_i^\top = A_i,$$

$$\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle \text{ is closed under multiplication},$$

then $(X, \{R_i\}_{i=0}^d)$ is called a symmetric association scheme, $\mathcal{A}$ is called its Bose-Mesner algebra.

Symmetric ssociation schemes

For a symmetric association scheme, the Bose-Mesner algebra

$$\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle$$

is simultaneously diagonalizable:

$$A_j \sim \begin{pmatrix} [p_{0j}]^{m_0} & & & \\ & [p_{1j}]^{m_1} & & \\ & & \ddots & \\ & & & [p_{d\,j}]^{m_d} \end{pmatrix} \to \quad P = (p_{ij})$$

eigenmatrix

Oriented cover

Triangular graph $T(n)$ $(n \geq 4)$:

$$V = \{\{i,j\} \mid i,j \in \{1,\dots,n\},\ i \neq j\},$$
$$R_1 = \{\{\{i,j\},\{k,l\}\} \mid |\{i,j\} \cap \{k,l\}| = 1\},$$
$$R_2 = \{\{\{i,j\},\{k,l\}\} \mid \{i,j\} \cap \{k,l\} = \emptyset\},$$

Let A_i be the adjacency matrix of R_i, for $i = 1,2$, and set $A_0 = I$. They form a symmetric association scheme.
Oriented (directed) version:

$$V = \{(i,j) \mid i,j \in \{1,\dots,n\},\ i \neq j\},$$
$$R_1 = \{((i,j),(k,l)) \mid i \neq j = k \neq l \neq i\}.$$

Then $R_2, R_3, \dots$?

Oriented cover

$$V = \{(i,j) \mid i,j \in \{1,\dots,n\},\ i \neq j\}.$$

$$\begin{aligned}
R_1 &= \{((i,j),(j,i)) \mid i \neq j\},\\
R_2 &= \{((i,j),(k,l)) \mid j \neq i = k \neq l \neq j\},\\
R_3 &= \{((i,j),(k,l)) \mid j \neq i = l \neq k \neq j\},\\
R_4 &= \{((i,j),(k,l)) \mid i \neq j = l \neq k \neq i\},\\
R_5 &= \{((i,j),(k,l)) \mid i \neq j = k \neq l \neq i\},\\
R_6 &= \{((i,j),(k,l)) \mid \{i,j\} \cap \{k,l\} = \emptyset\}.
\end{aligned}$$

Let A_i be the adjacency matrix of R_i, for $i = 1, 2$, and set $A_0 = I$. They form an association scheme (in a broad sense), i.e., non-symmetric, non-commutative.

Non-commutative association schemes

If X is a finite set,

$$X \times X = R_0 \cup R_1 \cup \cdots \cup R_d \quad \text{(partition)},$$

adjacency matrices $A_0, \ A_1, \ \ldots, \ A_d$

satisfy

$$A_0 = I,$$

$$\sum_{i=0}^{d} A_i = J \quad \text{(all-one matrix)},$$

$$\forall i, \exists i', \ A_i^\top = A_{i'},$$

$$\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle \text{ is closed under multiplication},$$

then $(X, \{R_i\}_{i=0}^d)$ is called a (non-commutative) association scheme, $\mathcal{A}$ is called its Bose-Mesner algebra.

Oriented cover

$$V = \{(i,j) \mid i,j \in \{1,\dots,n\},\ i \neq j\}.$$

$$\begin{aligned}
R_1 &= \{((i,j),(j,i)) \mid i \neq j\},\\
R_2 &= \{((i,j),(k,l)) \mid j \neq i = k \neq l \neq j\},\\
R_3 &= \{((i,j),(k,l)) \mid j \neq i = l \neq k \neq j\},\\
R_4 &= \{((i,j),(k,l)) \mid i \neq j = l \neq k \neq i\},\\
R_5 &= \{((i,j),(k,l)) \mid i \neq j = k \neq l \neq i\},\\
R_6 &= \{((i,j),(k,l)) \mid \{i,j\} \cap \{k,l\} = \emptyset\}.
\end{aligned}$$

Let A_i be the adjacency matrix of R_i, for $i = 1,2$, and set $A_0 = I$. They form a (non-commutative) association scheme. They cannot be simultaneously diagonalized. In fact,

$$\langle A_0,\dots,A_6\rangle \cong M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}),$$
$$7 = 1 + 2^2 + 1 + 1.$$

Two bases of the 7-dimensional algebra

$$
\begin{aligned}
&\langle A_0, \ldots, A_6 \rangle \\
&= \langle E_0 \rangle \oplus \langle E_1^{(1,1)}, E_1^{(1,2)}, E_1^{(2,1)}, E_1^{(2,2)} \rangle \oplus \langle E_2 \rangle \oplus \langle E_3 \rangle \\
&\cong M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C})
\end{aligned}
$$

Simultaneous block diagonalization:

$$
A_j \sim \begin{pmatrix} p_{0j} & & & \\ & \begin{pmatrix} p_{1j}^{(1,1)} & p_{1j}^{(1,2)} \\ p_{1j}^{(2,1)} & p_{1j}^{(2,2)} \end{pmatrix} & & \\ & & p_{2j} & \\ & & & p_{3j} \end{pmatrix}
$$

$$
A_j = p_{0j} E_0 + \sum_{k,l} p_{1j}^{(k,l)} E_1^{(k,l)} + p_{2j} E_2 + p_{3j} E_3 .
$$

Eigenmatrix of the oriented cover of $T(n)$

The jth column of the matrix P consists of the coefficients of A_j when written as a linear combination of E's.

$$\begin{array}{r} E_0 \\ E_1^{(1,1)} \\ E_1^{(1,2)} \\ E_1^{(2,1)} \\ E_1^{(2,2)} \\ E_2 \\ E_3 \end{array} \begin{bmatrix} 1 & 1 & n-2 & n-2 & n-2 & n-2 & (n-2)(n-3) \\ 1 & 1 & \frac{n-4}{2} & \frac{n-4}{2} & \frac{n-4}{2} & \frac{n-4}{2} & -2(n-3) \\ 0 & 0 & m & -m & -m & m & 0 \\ 0 & 0 & m & m & -m & -m & 0 \\ 1 & -1 & \frac{n-2}{2} & -\frac{n-2}{2} & \frac{n-2}{2} & -\frac{n-2}{2} & 0 \\ 1 & 1 & -1 & -1 & -1 & -1 & 2 \\ 1 & -1 & -1 & 1 & -1 & 1 & 0 \end{bmatrix}$$

where

$$m = \frac{\sqrt{n(n-2)}}{2}.$$

Directed strongly regular graph (Duval, 1988)

$$V = \{(i,j) \mid i,j \in \{1,\dots,n\},\ i \neq j\}.$$

$$R_1 = \{((i,j),(j,i)) \mid i \neq j\},$$
$$R_5 = \{((i,j),(k,l)) \mid i \neq j = k \neq l \neq i\}.$$

The matrix $A = A_1 + A_5$ satisfies

$$AJ = kJ,$$
$$A^2 = tI + \lambda A + \mu(J - I - A),$$

where

$$k = n-1, \quad t = 1, \quad \lambda = 0, \quad \mu = 1.$$

This is very similar to the property of the adjacency matrix of a SRG:

$$AJ = kJ,$$
$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Directed strongly regular graph

SRG:

$$AJ = kJ,$$
$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Definition

Let Γ be a directed graph with adjacency matrix A. Then Γ is called a directed strongly regular graph (DSRG) with parameters (k, μ, λ, t) if

$$AJ = kJ,$$
$$A^2 = tI + \lambda A + \mu(J - I - A).$$

Note $0 \leq t \leq k$, and

$$t = k \iff \text{SRG}$$
$$t = 0 \iff \text{tournament.}$$

Directed strongly regular graph

Assume

$$AJ = kJ,$$
$$A^2 = tI + \lambda A + \mu(J - I - A),$$
$$0 < t < k.$$

Theorem (Klin-M.-Muzychuk-Zieschang 2004)

The adjacency matrix A cannot be contained in the Bose-Mesner algebra of a commutative association scheme. In particular, algebra generated by A under $\cdot, \circ$ has dimension at least 6.

2-$(v,k,1)$ design

Definition

A 2-(v,k,λ) design is an incidence structure $(\mathcal{P},\mathcal{B})$, where $\mathcal{B}\subset\binom{\mathcal{P}}{k}$, and every pair $i,j\in\mathcal{P}$ is contained in λ members of $\mathcal{B}$.

Assume $(\mathcal{P},\mathcal{B})$ is a 2-$(v,k,1)$ design, and set

$$\mathcal{F}=\{(i,B)\in\mathcal{P}\times\mathcal{B}\mid x\in B\}.$$

Example: $\mathcal{B}=\binom{\mathcal{P}}{2}$, $\mathcal{P}=\{1,\dots,n\}$. Then

$$\mathcal{F}=\{(i,\{i,j\})\mid i,j\in\{1,\dots,n\},\ i\neq j\}$$

which corresponds bijectively to

$$V=\{(i,j)\mid i,j\in\{1,\dots,n\},\ i\neq j\}.$$

2-$(v,k,1)$ design

Assume $(\mathcal{P},\mathcal{B})$ is a 2-$(v,k,1)$ design, and set

$$\mathcal{F} = \{(i,B) \in \mathcal{P} \times \mathcal{B} \mid x \in B\},$$

$$\begin{aligned}
R_1 &= \{((i,B),(j,B)) \mid i \neq j\},\\
R_2 &= \{((i,B),(i,B')) \mid B \neq B'\},\\
R_3 &= \{((i,B),(j,B')) \mid j \neq i \in B'\},\\
R_4 &= \{((i,B),(j,B')) \mid i,j \notin B \cap B' \neq \emptyset\},\\
R_5 &= \{((i,B),(j,B')) \mid i \neq j \in B\},\\
R_6 &= \{((i,B),(j,B')) \mid B \cap B' = \emptyset\}.
\end{aligned}$$

Let A_i be the adjacency matrix of R_i, for $i = 1, 2$, and set $A_0 = I$. The flag algebra is (Klin-M.-Muzychuk-Zieschang, 2004):

$$\langle A_0, \dots, A_6 \rangle \cong M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}).$$

The end

Problem

- Generalize our result on the calculation of the eigenmatrix for the oriented $T(n)$, to that of the flag algebra of a 2-$(v, k, 1)$ design.
- Find other classes of DSRG for which eigenmatrix can be calculated. Is the eigenmatrix determined by (k, μ, λ, t)?

Tomorrow, I will give another talk at China University of Geoscience in Beijing:

> *Quasi-symmetric* 2-$(56, 16, 18)$ *designs constructed from the dual of the quasi-symmetric* 2-$(21, 6, 4)$ *design as a Hoffman coclique*

where I will describe another relationship between strongly regular graphs and 2-designs. Thank you very much for your attention.