Quasi-symmetric 2-$(56, 16, 18)$ designs constructed from the dual of the quasi-symmetric 2-$(21, 6, 4)$ design as a Hoffman coclique

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t-(v,k,λ) designs

A t-(v,k,λ) design is a pair $(\mathcal{P},\mathcal{B})$, where

- $|\mathcal{P}| = v$, $\mathcal{B} \subset \binom{\mathcal{P}}{k}$,
- $|\{B \in \mathcal{B} \mid B \supset T\}| = \lambda$ for all $T \in \binom{\mathcal{P}}{t}$.

Elements of $\mathcal{P}$ are called points, and elements of $\mathcal{B}$ are called blocks, or lines.

If $t = 2$ and $\lambda = 1$, then the second condition is

$$|\{B \in \mathcal{B} \mid B \ni p, q\}| = 1 \text{ for all } p, q \in \mathcal{P},\ p \neq q.$$

which can be rephrased as

every pair of distinct points lie in a unique line.

$(t+1)$-design $\implies$ t-design.

2-(v,k,λ) designs

A 2-(v,k,λ) design is a pair $(\mathcal{P},\mathcal{B})$, where

- $|\mathcal{P}| = v$, $\mathcal{B} \subset \binom{\mathcal{P}}{k}$,
- $|\{B \in \mathcal{B} \mid B \ni p, q\}| = \lambda$ for all $\{p,q\} \in \binom{\mathcal{P}}{2}$.

$$\text{symmetric if } |\{|B \cap B'| \mid B, B' \in \mathcal{B},\ B \neq B'\}| = 1,$$
$$\text{quasi-symmetric if } |\{|B \cap B'| \mid B, B' \in \mathcal{B},\ B \neq B'\}| = 2.$$

For a symmetric design, we have

$$\{|B \cap B'| \mid B, B' \in \mathcal{B},\ B \neq B'\} = \{\lambda\}.$$

For a quasi-symmetric design, write

$$\{|B \cap B'| \mid B, B' \in \mathcal{B},\ B \neq B'\} = \{x, y\}$$

with $x < y$ (intersection numbers, uniquely determined by v, k, λ).

A. Neumaier (1982)

1. 2-$(v, k, 1)$ designs, $x = 0$, $y = 1$.
2. Hadamard 3-design, 2-$(4n, 2n, 2n-1)$, $x = 0$, $y = n$; more generally, resolvable designs ($x = 0$)
3. residual of biplanes (finitely many known)

Other examples:

1. Exceptional designs: not in the above classes.
2. 4-$(23, 7, 1)$ design or its complement is the only quasi-symmetric design which is a 4-design. This design has automorphism group M_{23}, a sporadic finite simple group discovered by Mathieu (1873). The uniqueness of such a design is due to Witt (1938).

Designs related to Mathieu groups

name	parameters	int. numbers
W_{24}	$5\text{-}(24,8,1)$	$4,2,0$
W_{23}	$4\text{-}(23,7,1)$	$3,1$
W_{22}	$3\text{-}(22,6,1)$	$2,0$

These designs are unique, without assumption on intersection numbers (Witt 1938).

The design W_{22} gives rise to

parameters	int. numbers
$2\text{-}(21,6,4)$	$2,0$

This design is unique as a quasi-symmetric design (Tonchev 1986). However, according to Martinjak and Pavčević (2009), there are at least 1,700,745 $2\text{-}(21,6,4)$ designs.
$\rightarrow$ Table of exceptional quasi-symmetric designs in "CRC Handbook of Combinatorial Designs" (2007).

2-$(56, 16, 18)$ $(x = 4,\ y = 8)$

The existence of a quasi-symmetric 2-$(56, 16, 18)$ design was unknown until:

Theorem (Krčadinac–Vlahović, 2016)

There are at least 3 non-isomorphic quasi-symmetric 2-$(56, 16, 18)$ designs.

These were found by computer under the assumption that the automorphism group contains $\mathbb{Z}_2^5.A_5$ of order 960.
One of the 3 designs has full automorphism group $M_{21}.\mathbb{Z}_2 = L_3(4).2$, which is the automorphism group of the unique 2-$(21, 6, 4)$ design.

2-$(56, 16, 18)$ and 2-$(21, 6, 4)$

The incidence matrix of a design $(\mathcal{P}, \mathcal{B})$ is the $|\mathcal{P}| \times |\mathcal{B}|$ matrix M whose (p, B)-entry is

$$M_{p,B} = \begin{cases} 1 & \text{if } p \in B, \\ 0 & \text{otherwise,} \end{cases}$$

where $p \in \mathcal{P}$, $B \in \mathcal{B}$.

2-$(21, 6, 4)$ design $\rightarrow$ 21×56 matrix M

2-$(56, 16, 18)$ design $\rightarrow$ 56×231 matrix $N^\top$

$\rightarrow$ 231×56 matrix N

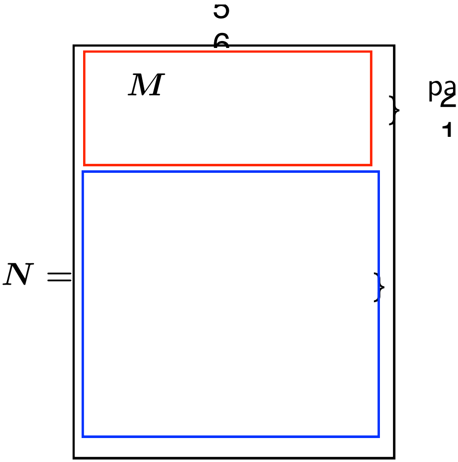
Does N contain M as a submatrix?

$$NN^\top = 16I + 8A + 4(J - I - A), \qquad (231)$$

$$MM^\top = 16I + 4(J - I), \qquad (21)$$

because intersection numbers are $4, 8$.

2-(56, 16, 18) and 2-(21, 6, 4)



pairwise inner product 4

210 rows.

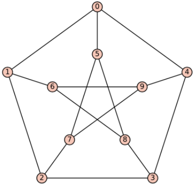
Strongly regular graphs

Definition

A graph Γ is called a strongly regular graph (SRG) with parameters (k, λ, μ) if

- valency is k,
- each edge is contained in λ triangles,
- each pair of non-adjacent vertices has μ common neighbors.

The Petersen graph is a SRG with parameters $(3, 0, 1)$.



Algebraic Graph Theory

For a regular graph Γ, the following are equivalent:

- Γ is strongly regular,
- the adjacency matrix has 3 distinct eigenvalues.

Theorem (See Brouwer-Haemers, Theorem 3.5.2)

Let Γ be a k-regular graph on n vertices, with smallest eigenvalue θ. Then every coclique of Γ has size at most

$$n\frac{-\theta}{k-\theta}.$$

A coclique whose size achieve the upper bound is called a Hoffman coclique.
Vertices outside a Hoffman coclique C has a constant number of neighbors in C (equitable partition).

Quasi-symmetric design $\to$ SRG

Theorem

Let $(\mathcal{P}, \mathcal{B})$ be a quasi-symmetric design with intersection numbers $x < y$. Define

$$E = \{\{B_1, B_2\} \mid B_1, B_2 \in \mathcal{B},\ |B_1 \cap B_2| = y\}.$$

Then $(\mathcal{B}, E)$ is a strongly regular graph.

The graph obtained in this way is called the block graph of the quasi-symmetric design.
If N is the transpose of the incidence matrix, then the adjacency matrix A of the block graph can be expressed as:

$$NN^\top = kI + yA + x(J - I - A).$$

Quasi-symmetric design $\rightarrow$ SRG

$\mathbf{2}$-$(\mathbf{56}, \mathbf{16}, \mathbf{18})$ design $\quad\rightarrow\quad$ $\mathbf{231} \times \mathbf{56}$ matrix $\boldsymbol{N}$

$$\boldsymbol{N}\boldsymbol{N}^{\top} = \mathbf{16}\boldsymbol{I} + \mathbf{8}\boldsymbol{A} + \mathbf{4}(\boldsymbol{J} - \boldsymbol{I} - \boldsymbol{A}), \tag{231}$$

because intersection numbers are $\mathbf{4}, \mathbf{8}$.
The $(\mathbf{0}, \mathbf{1})$-matrix $\boldsymbol{A}$ is the adjacency matrix of a SRG on $\mathbf{231}$ vertices with parameters $(\mathbf{30}, \mathbf{9}, \mathbf{3})$ and smallest eigenvalue $-\mathbf{3}$. The bound is

$$\boldsymbol{n}\frac{-\boldsymbol{\theta}}{\boldsymbol{k} - \boldsymbol{\theta}} = \mathbf{231}\frac{-(-\mathbf{3})}{\mathbf{30} - (-\mathbf{3})} = \mathbf{21}.$$

So

$$\boldsymbol{M}\boldsymbol{M}^{\top} = \mathbf{16}\boldsymbol{I} + \mathbf{4}(\boldsymbol{J} - \boldsymbol{I}), \tag{21}$$

will correspond to a Hoffman coclique.

From $(21, 6, 4)$ to $(56, 16, 18)$

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be the unique quasi-symmetric 2-$(21, 6, 4)$ design. The dual $\overline{\mathcal{D}} = (\mathcal{B}, \overline{\mathcal{P}})$ is

$$\overline{\mathcal{P}} = \{[p] \mid p \in \mathcal{P}\}, \quad |\overline{\mathcal{P}}| = 21,$$

where

$$[p] = \{B \in \mathcal{B} \mid p \in B\} \quad (p \in \mathcal{P}).$$

Define

$$\mathcal{Q} = \{Q \in \binom{\mathcal{P}}{5} \mid |Q \cap B| \leq 2 \ (\forall B \in \mathcal{B})\}, \quad |\mathcal{Q}| = 21,$$

$$\mathcal{R} = \{([p_1] \cup [p_2] \cup [p_3]) \triangle ([p_4] \cup [p_5]) \mid \{p_1, \dots, p_5\} \in \mathcal{Q}\}$$
$$\subseteq \binom{\mathcal{B}}{16}, \quad |\mathcal{R}| = 210.$$

Then $(\mathcal{B}, \overline{\mathcal{P}} \cup \mathcal{R})$ is a quasi-symmetric 2-$(56, 16, 18)$ design.

The end

Problem

- Classify quasi-symmetric 2-$(56, 16, 18)$ designs constructible from the quasi-symmetric 2-$(21, 6, 4)$ designs as described above. Are there more than 2 designs up to isomorphism?
- Classify quasi-symmetric 2-$(56, 16, 18)$ designs. Are there more than 3 designs up to isomorphism?
- Classify quasi-symmetric 2-$(56, 16, 6)$ designs. Is there a design constructible from the quasi-symmetric 2-$(21, 6, 4)$ designs in a similar manner as above?

Thank you very much for your attention.