The Reed-Muller code RM(1,4), the Barnes-Wall lattice BW(16), and graphs with smallest eigenvalue -3

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A warning about the term "lattice"

A lattice could mean:

- a partially ordered set with unique least upper bounds and greatest lower bounds, or
- $\mathbb{Z}^n \subset \mathbb{R}^n$, or
- a subgroup $L \subset \mathbb{R}^n$ generated by a basis

In this talk, a lattice will mean the third variant.

$$L \cong \mathbb{Z}^n \quad \text{as abstract groups}$$

L may not be isometric to $\mathbb{Z}^n$.

Vector representation of a graph

By a representation of a graph, we mean

$$\{\text{vertices}\} \to \begin{array}{l} \boldsymbol{L} \\ \text{(fixed distance from } \mathbf{0}\text{)} \end{array}$$

such that, for two distinct vertices $\boldsymbol{u}, \boldsymbol{v}$,

$$\begin{aligned} \boldsymbol{u} \sim \boldsymbol{v} &\iff (\boldsymbol{u}, \boldsymbol{v}) = \mathbf{1}, \\ \boldsymbol{u} \not\sim \boldsymbol{v} &\iff (\boldsymbol{u}, \boldsymbol{v}) = \mathbf{0}. \end{aligned}$$

Vector representation of a graph (Example)

$\boldsymbol{L} = \mathbb{Z}^{\boldsymbol{n}}$. Vertices are

$$(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$$

Edges are

$$u \sim v \iff \begin{array}{l} u = (\dots \quad 1 \quad \dots \quad {\color{red}1} \quad \dots \quad \dots \quad \dots) \\ v = (\dots \quad \dots \quad \dots \quad {\color{red}1} \quad \dots \quad 1 \quad \dots) \end{array}$$

This is just a line graph of a graph on n vertices.
How do we distinguish line graphs from non-line graphs?
(orthonormal basis, vectors of norm $2\dots$)

Vector representation of a graph (a formal definition)

Let (G, E) be a graph, m a positive integer. A mapping

$$\varphi : V(G) \to \mathbb{R}^n$$

is a representation of norm m if

$$(\varphi(u), \varphi(v)) = \begin{cases} m & \text{if } u = v, \\ 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $L(K_n)$ has a representation of norm 2.

$$\exists\varphi \text{ of norm } m \iff A(G) + mI \text{ is positive semidefinite}$$
$$\iff \lambda_{\min}(G) \geq -m.$$

Vector representation and the lattice

Let (G, E) be a graph, m a positive integer. Assume $\lambda_{\min}(G) \geq -m$. Let

$$\varphi : V(G) \to \mathbb{R}^n$$

be a representation of norm m. Then

$$L = \{\mathbb{Z}\text{-linear combinations of } \varphi(V(G))\}.$$

is a lattice. The dual of L is

$$L^* = \{y \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \ (\forall x \in L)\} \supset L.$$

If G is a line graph, then L^* contains an orthonormal basis. Define

$$\mu_m^*(G) = \min L^* = \min\{(y, y) \mid y \in L^*, \ y \neq 0\}.$$

Minimum of the dual lattice

Assume $\lambda_{\min}(G) \geq -m$.

$$\mu_m^*(G) = \min\{(y, y) \mid y \in L^*,\ y \neq 0\},$$

where L is the lattice generated by a norm m representation of G.

Proposition

If G is a line graph, then $\mu_2^*(G) \leq 1$.

If $|V(G)| \leq 5$ and $\lambda_{\min}(G) \geq -2$, then $\mu_2^*(G) \leq 1$.
However,

$$\mu_2^*(E_6) = \frac{4}{3} > 1.$$

$\mu_2^*(G)$ and $\mu_3^*(G)$

$\|V(G)\|$	$\mu_2^*(G)$
≤ 5	≤ 1
E_6	$4/3$
≤ 7	< 2
E_8	2

$\|V(G)\|$	$\mu_3^*(G)$
≤ 8	≤ 1
9	$8/7, 16/15$
?	?
16	2
23	3

There exists a graph G with 16 vertices such that $\mu_3^*(G) = 2$. Its norm 3 representation generates the overlattice of the Barnes-Wall lattice.

$RM(1,4)$

The Reed-Muller code $C = RM(1,4)$ is the 5-dimensional subspace of $\mathbb{F}_2^{16}$ whose basis is

$$\begin{array}{c}
1111111100000000\\
1111000011110000\\
1100110011001100\\
1010101010101010\\
1111111111111111
\end{array}$$

Consider

$$\pi : \mathbb{Z}^{16} \to \mathbb{F}_2^{16} \quad \text{(reducing modulo } 2\text{)}$$

and set

$$\Lambda = \frac{1}{\sqrt{2}}\pi^{-1}(C) \subset \mathbb{R}^{16}.$$

The Barnes-Wall lattice $BW(16)$

$$
\begin{aligned}
C &= RM(1,4),\\
\pi &: \mathbb{Z}^{16} \to \mathbb{F}_2^{16} \quad \text{(reducing modulo 2)},\\
\Lambda &= \frac{1}{\sqrt{2}}\pi^{-1}(C) \subset \mathbb{R}^{16},\\
u &= \frac{1}{\sqrt{2}}(1,1,\dots,1) \in \Lambda,\\
BW(16) &= \{x \in \Lambda \mid (x,u) \equiv 0 \pmod 2\}.
\end{aligned}
$$

Then there exists $v \in BW(16)^*$ with $(v,v) = 3$.
The overlattice of the Barnes-Wall lattice is

$$BW(16) + \mathbb{Z}v.$$

$\mu_3^*(G)$

$\lvert V(G)\rvert$	$\mu_3^*(G)$
≤ 8	≤ 1
9	$8/7, 16/15$
$?$	$?$
16 (min ?)	2
23 (min ?)	3

- $\exists G$ with 16 vertices such that $\mu_3^*(G) = 2$, its norm 3 representation generates the overlattice of the Barnes-Wall lattice.
- $\exists G$ with 23 vertices such that $\mu_3^*(G) = 3$, its norm 3 representation generates a sublattice of index 2 in the shorter Leech lattice.