A graph with smallest eigenvalue -3 related to the shorter Leech lattice

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A warning about the term "lattice"

A lattice could mean:

- a partially ordered set with unique least upper bounds and greatest lower bounds, or
- $\mathbb{Z}^n \subset \mathbb{R}^n$, or
- a subgroup $L \subset \mathbb{R}^n$ generated by a basis

In this talk, a lattice will mean the third variant.

$$L \cong \mathbb{Z}^n \quad \text{as abstract groups}$$

L may not be isometric to $\mathbb{Z}^n$.

Vector representation of a graph

By a representation of a graph, we mean

$$\{\text{vertices}\} \to \begin{array}{l} \boldsymbol{L} \\ \text{(fixed distance from } \mathbf{0}) \end{array}$$

such that, for two distinct vertices $\boldsymbol{u}, \boldsymbol{v}$,

$$\begin{aligned} \boldsymbol{u} \sim \boldsymbol{v} &\iff (\boldsymbol{u}, \boldsymbol{v}) = \mathbf{1}, \\ \boldsymbol{u} \not\sim \boldsymbol{v} &\iff (\boldsymbol{u}, \boldsymbol{v}) = \mathbf{0}. \end{aligned}$$

Vector representation of a graph (Example)

$\boldsymbol{L} = \mathbb{Z}^{\boldsymbol{n}}$. Vertices are

$$(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$$

Edges are

$$u \sim v \iff \begin{array}{l} u = (\dots \quad 1 \quad \dots \quad \color{red}{1} \quad \dots \quad \dots \quad \dots) \\ v = (\dots \quad \dots \quad \dots \quad \color{red}{1} \quad \dots \quad 1 \quad \dots) \end{array}$$

This is just a line graph of a graph on n vertices.
How do we distinguish line graphs from non-line graphs?
(orthonormal basis, vectors of norm 2. . .)

Vector representation of a graph (a formal definition)

Let (G, E) be a graph, m a positive integer. A mapping

$$\varphi : V(G) \to \mathbb{R}^n$$

is a representation of norm m if $\varphi(V(G))$ spans $\mathbb{R}^n$ and

$$(\varphi(u), \varphi(v)) = \begin{cases} m & \text{if } u = v, \\ 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $L(K_n)$ has a representation of norm 2 in $\mathbb{R}^n$.

$$\begin{aligned} \exists\varphi \text{ of norm } m \iff & A(G) + mI \text{ is positive semidefinite} \\ \iff & \lambda_{\min}(G) \geq -m. \end{aligned}$$

Vector representation and the lattice

Let (G, E) be a graph, m a positive integer. Assume $\lambda_{\min}(G) \geq -m$. Let

$$\varphi : V(G) \to \mathbb{R}^n$$

be a representation of norm m. Then

$$L = \{\mathbb{Z}\text{-linear combinations of } \varphi(V(G))\}.$$

is a lattice.
The lattice L is unique up to isometry (independent of φ).

Dual lattice

The dual of a lattice L is

$$L^* = \{y \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z}\ (\forall x \in L)\} \supset L.$$

If G is the line graph of a graph H, $\varphi : V(G) \to L \to \mathbb{R}^n$ is a representation of norm 2

$$\Longrightarrow \begin{cases} \exists \psi : V(G) \to \mathbb{R}^{|V(H)|} : \text{ representation of norm } 2, \\ (\psi(V(G)), e_i) \in \{0, 1\} \quad (1 \le i \le |V(H)|) \end{cases}$$

$$\Longrightarrow \begin{cases} \exists \text{ embedding } L \to \mathbb{R}^{|V(H)|}, \\ L^* \text{ contains vectors of norm at most } 1. \end{cases}$$

The minimum norm of the dual L^* of the lattice L generated by a representation of a graph gives an important information about how close G is to a line graph.

Minimum of the dual lattice

Assume $\lambda_{\min}(G) \geq -m$.

$$\mu_m^*(G) = \min\{(y, y) \mid y \in L^*,\ y \neq 0\},$$

where L is the lattice generated by a norm m representation of G.

Proposition

If G is a line graph, then $\mu_2^*(G) \leq 1$.

If $|V(G)| \leq 5$ and $\lambda_{\min}(G) \geq -2$, then $\mu_2^*(G) \leq 1$.
However,

$$\mu_2^*(E_6) = \frac{4}{3} > 1.$$

$\mu_2^*(G)$ and $\mu_3^*(G)$

$\lvert V(G)\rvert$	$\mu_2^*(G)$
≤ 5	≤ 1
E_6	$4/3$
≤ 7	< 2
E_8	2

$\lvert V(G)\rvert$	$\mu_3^*(G)$
≤ 8	≤ 1
9	$8/7, 16/15$
?	?
16	2
23	3

There exists a graph G with 16 vertices such that $\mu_3^*(G) = 2$. Its norm 3 representation generates the overlattice of the Barnes-Wall lattice.

$\mu_3^*(G)$

$\lvert V(G) \rvert$	$\mu_3^*(G)$
≤ 8	≤ 1
9	$8/7, 16/15$
?	?
16 (min ?)	2
23 (min ?)	3

- $\exists G$ with 16 vertices such that $\mu_3^*(G) = 2$, its norm 3 representation generates the overlattice of the Barnes-Wall lattice.
- $\exists G$ with 23 vertices such that $\mu_3^*(G) = 3$, its norm 3 representation generates the shorter Leech lattice.

The shorter Leech lattice

Characterized by

- unimodular
- rank 23
- minimum norm $\mathbf{3}$

Its kissing number (the number of norm $\mathbf{3}$ vectors) is $\mathbf{4600}$.

The (complement of the) McLaughlin graph

Unique strongly regular graph with parameters

$$v = 275,\ k = 162, \lambda = 105, \mu = 81.$$

It has smallest eigenvalue -3 with multiplicity 252

$$\mathrm{rank}(A + 3I) = 23$$

$\implies$ a lattice of rank 23 generated by norm 3 vectors.
This lattice is not the shorter Leech lattice, rather, it is a sublattice of index 3 in the shorter Leech lattice, with kissing number 550.

0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1	0	1	1
1	0	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	0	0	1	0
1	1	0	1	1	1	1	1	1	1	1	1	0	0	1	1	0	1	1	1	0	0	0
1	1	1	0	1	1	1	1	1	1	1	1	0	1	0	1	1	0	1	0	0	1	0
1	1	1	1	0	1	1	1	1	1	1	1	0	1	0	1	0	0	0	1	1	1	0
1	1	1	1	1	0	1	1	1	1	1	1	0	1	1	0	1	1	1	1	1	1	0
1	1	1	1	1	1	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	0	0
1	1	1	1	1	1	1	0	1	1	1	1	0	1	1	1	0	0	1	1	1	0	0
1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	0	1	1	0	1	0	0	1
1	1	1	1	1	1	1	1	1	0	1	1	0	1	1	0	0	0	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	0	1	0	0	0	1	0
1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1	1	0	0
1	1	0	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1	1
1	1	1	0	0	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	0
1	1	1	1	1	0	0	1	0	0	1	1	1	1	1	0	1	1	1	1	0	1	0
0	0	0	1	0	1	0	0	1	0	0	1	1	1	1	1	0	1	1	1	0	1	0
1	1	1	0	0	1	1	0	1	0	1	1	1	1	1	1	1	0	1	1	0	0	1
1	1	1	1	0	1	1	1	0	1	0	1	1	1	1	1	1	1	0	1	1	1	0
1	0	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0	1	0	1
0	0	0	0	1	1	1	1	0	1	0	1	1	0	0	0	0	0	1	1	0	0	0
1	1	0	1	1	1	0	0	0	1	1	0	0	1	1	1	1	0	1	0	0	0	0
1	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	1	0	1	0	0	0