Complementary Ramsey Numbers and Ramsey Graphs

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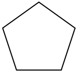
Ramsey Numbers

For a graph G,

$\alpha(G) =$ independence number $= \max\{\#\text{independent set}\}$
$\omega(G) =$ clique number $= \max\{\#\text{clique}\}$



$$\omega(C_5) = \alpha(C_5) = 2.$$

$$\forall G \text{ with 6 vertices, } \omega(G) \geq 3 \text{ or } \alpha(G) \geq 3.$$

These facts can be conveniently described by the Ramsey number:

$$R(3,3) = 6.$$

The smallest number of vertices required to guarantee $\omega \geq 3$ or $\alpha \geq 3$ (precise definition in the next slide).

Ramsey Numbers and a Generalization

Definition

The Ramsey number $R(m_1, m_2)$ is defined as:

$$
\begin{aligned}
&R(m_1, m_2) \\
&= \min\{n \mid |V(G)| = n \implies \omega(G) \geq m_1 \text{ or } \alpha(G) \geq m_2\} \\
&= \min\{n \mid |V(G)| = n \implies \omega(G) \geq m_1 \text{ or } \omega(\overline{G}) \geq m_2\}
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A graph with n vertices defines a partition of $E(K_n)$ into 2 parts, "edges" and "non-edges".

Generalized Ramsey numbers $R(m_1, m_2, \ldots, m_k)$ can be defined if we consider partitions of $E(K_n)$ into k parts, i.e., (not necessarily proper) edge-colorings.

Definition (Complementary Ramsey numbers)

We write by $[n] = \{1, 2, \ldots, n\}$, and denote by $E(K_n) = \binom{[n]}{2}$ the set of 2-subsets of $[n]$. The set of k-edge-coloring of K_n is denoted by $C(n, k)$:

$$C(n,k) = \{f \mid f : E(K_n) \to [k]\}.$$

We abbreviate

$$\omega_i(f) = \omega([n], f^{-1}(i)), \quad \alpha_i(f) = \alpha([n], f^{-1}(i)).$$

$$R(m_1, \ldots, m_k) = \min\{n \mid \forall f \in C(n,k),\ \exists i \in [k],\ \omega_i(f) \geq m_i\}$$

$$\bar{R}(m_1, \ldots, m_k) = \min\{n \mid \forall f \in C(n,k),\ \exists i \in [k],\ \alpha_i(f) \geq m_i\}$$

The last one is called the complemtary Ramsey number.

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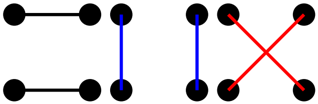
$$\bar{R}(m_1, m_2) = R(m_2, m_1) = R(m_1, m_2).$$

So we focus on the case $k = 3$. Also we assume $m_i \geq 3$.

$\bar{R}(3,3,3) = 5$ by factorization



- K_4 has a 3-edge-coloring f into $2K_2$ (a 1-factorization). Then $\alpha_i(f) = 2$ for $i = 1, 2, 3$. This implies $\bar{R}(3,3,3) > 4$.
- If f is a 3-edge-coloring of K_5, then some color i has at most 3 edges, so $\alpha_i(f) \geq 3$. This implies $\bar{R}(3,3,3) \leq 5$.

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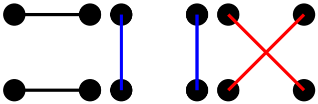

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m	3	4	5	6	7	8	9–13	14–
$\bar{R}(m,3,3)$	5	5	5	6	6	6	6	6
$\bar{R}(m,4,3)$	5	7	8	8	9	9	9	9
$\bar{R}(m,5,3)$	5	8	9	11	12	12	13	14
$\bar{R}(m,6,3)$	6	8	11	?	?	?	?	?
$\bar{R}(m,4,4)$	5	10	?	?	?	?	?	?

These were determined by Chung and Liu (1978).

History

- First considered by Erdős, Hajnal and Rado (1965).
- Erdős and Szemerédi (1972) gave an asymptotic upper bound.
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Despite these work, some small complementary Ramsey numbers seem undetermined, for example,

$$\bar{R}(4,4,4) = 10, \quad \bar{R}(5,4,4) = ?$$

Our results

m	3	4	5–6	7	8–10	11–16	17	18–
$\bar{R}(m,4,4)$	5	10	13	14	15	16	17	18

Greenwood and Gleason (1955): $R(4,4) = 18$.

m	3	4	5	6	7	8	9–15	16–
$\bar{R}(m,6,3)$	6	8	11	13	14	16	17	18

Kéry (1964), Cariolaro (2007): $R(6,3) = 18$.

Ramsey (s, t)-graph

A graph G is said to be a Ramsey (s, t)-graph if

$$\omega(G) < s \text{ and } \alpha(G) < t.$$

We denote by $\mathcal{R}_n(s, t)$ the set of Ramsey (s, t)-graphs on the vertex-set $[n]$.

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B.D. McKay has database of (known) Ramsey graphs.

$$\begin{aligned} |\mathcal{R}_{18}(4,4)| &= 0, \\ |\mathcal{R}_{17}(4,4)| &= 1, \\ &\vdots \\ |\mathcal{R}_{12}(4,4)| &= 1449166, \\ &\vdots \end{aligned}$$

From Ramsey (4, 4)-graphs to $\bar{R}(m, 4, 4)$

Lemma

Let

$$a_n = \min\{\alpha(G - H) \mid G, H \in \mathcal{R}_n(4,4),\ G \supseteq H\}.$$

Then

$$\bar{R}(m, 4, 4) = 1 + \max\{n \in \mathbb{N} \mid a_n < m\}.$$

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Note

$$\begin{aligned} a_n = \min\{\alpha(G-H) \mid\ & G:\ \text{maximal in } \mathcal{R}_n(4,4), \\ & H:\ \text{minimal in } \mathcal{R}_n(4,4), \\ & G \supseteq H\} \end{aligned}$$

Database gives graphs only up to isomorphism.

Our method

We found an algorithm to determine:

$$\text{given } m \in \mathbb{N}, \quad G: \text{ maximal in } \mathcal{R}_n(4,4),$$

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Thank you very much for your attention!