Binary linear codes and designs

Akihiro Munemasa

Graduate School of Information Sciences
Tohoku University

November 8, 2018
Institut Teknologi Bandung

A conference graph

Let $X = \mathbb{F}_p$, where p is a prime with $p \equiv 1 \pmod 4$. Let Q be the set of quadratic residues in X. Consider the graph G with vertex set X, and edge set

$$E = \{(x, y) \mid x, y \in X,\ x - y \in Q\}.$$

Then for $x, y \in X$,

$$\#\{\text{common neighbors of } x, y\} = \begin{cases} \frac{p-1}{2} & \text{if } x = y, \\ \frac{p-5}{4} & \text{if } (x, y) \in E, \\ \frac{p-1}{4} & \text{otherwise.} \end{cases}$$

A graph on p (not necessarily prime) vertices satisfying the above condition is called a conference graph.
It is unknown whether a conference graph on 65 vertices exists.

Strongly regular graphs

A graph with edge set E is called a strongly regular graph with parameters (k, λ, μ) if

$$\#\{\text{common neighbors of } x, y\} = \begin{cases} k & \text{if } x = y, \\ \lambda & \text{if } (x, y) \in E, \\ \mu & \text{otherwise.} \end{cases}$$

A conference graph on p vertices is a strongly regular graph with parameters

$$(k, \lambda, \mu) = \left(\frac{p-1}{2}, \frac{p-5}{4}, \frac{p-1}{4} \right).$$

Adjacency matrix

The adjacency matrix A of a graph with vertex set X is the matrix whose rows and columns are indexed by X, and whose entries are defined by

$$A_{xy} = \begin{cases} 1 & \text{if } (x, y) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Adjacency matrix

The adjacency matrix A of a graph with vertex set X is the matrix whose rows and columns are indexed by X, and whose entries are defined by

$$A_{xy} = \begin{cases} 1 & \text{if } (x, y) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(A^2)_{xy} = \#\{\text{common neighbors of } x, y\}$, the condition

$$\#\{\text{common neighbors of } x, y\} = \begin{cases} k & \text{if } x = y, \\ \lambda & \text{if } (x, y) \in E, \\ \mu & \text{otherwise.} \end{cases}$$

tranlates to

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

where J is the "all-one" matrix.

The spectrum of the adjacency matrix

Since

$$(A^2)_{xx} = \#\{\text{neighbors of } x\} = k,$$

we have

$$AJ = JA = kJ.$$

Together with the equation

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

we can see that A is a root of a cubic polynomial.

Exercise: Show that A has exactly three distinct eigenvalues, provided $\mu \neq 0$. Express the three eigenvalues of A in terms of k, λ, μ.

A symmetric design

Let $X = \mathbb{F}_p$, where p is a prime of the form $p = 4m - 1$, where $m \in \mathbb{Z}$. Let Q be the set of quadratic residues in X. Then $|Q| = 2m - 1$. Let

$$\mathcal{B} = \{Q + a \mid a \in \mathbb{F}_p\}.$$

Then for any distinct $x, y \in K$,

$$|\{B \in \mathcal{B} \mid x, y \in B\}| = \frac{p-3}{4} = m - 1.$$

A family $\mathcal{B}$ of $(2m-1)$-element subsets of an $(4m-1)$-element set is called a symmetric $(4m-1, 2m-1, m-1)$ design if the above condition is satisfied.

A symmetric design

Let $X = \mathbb{F}_p$, where p is a prime of the form $p = 4m - 1$, where $m \in \mathbb{Z}$. Let Q be the set of quadratic residues in X. Then $|Q| = 2m - 1$. Let

$$\mathcal{B} = \{Q + a \mid a \in \mathbb{F}_p\}.$$

Then for any distinct $x, y \in K$,

$$|\{B \in \mathcal{B} \mid x, y \in B\}| = \frac{p-3}{4} = m - 1.$$

A family $\mathcal{B}$ of $(2m-1)$-element subsets of an $(4m-1)$-element set is called a symmetric $(4m-1, 2m-1, m-1)$ design if the above condition is satisfied.

Definition of 2-design

Let X be a finite set of v elements. Let $\mathcal{B}$ be a family of k-element subsets of X. If, for any distinct $x, y \in X$,

$$|\{B \in \mathcal{B} \mid x, y \in B\}| = \lambda,$$

then the pair $(X, \mathcal{B})$ is called a 2-(v, k, λ) design.

Definition of 2-design

Let X be a finite set of v elements. Let $\mathcal{B}$ be a family of k-element subsets of X. If, for any distinct $x, y \in X$,

$$|\{B \in \mathcal{B} \mid x, y \in B\}| = \lambda,$$

then the pair $(X, \mathcal{B})$ is called a 2-(v, k, λ) design.

Complete design: $\mathcal{B} = \{B \subset X \mid |B| = k\}$. $\lambda =?$

Definition of 2-design

Let X be a finite set of v elements. Let $\mathcal{B}$ be a family of k-element subsets of X. If, for any distinct $x, y \in X$,

$$|\{B \in \mathcal{B} \mid x, y \in B\}| = \lambda,$$

then the pair $(X, \mathcal{B})$ is called a 2-(v, k, λ) design.

Complete design: $\mathcal{B} = \{B \subset X \mid |B| = k\}$. $\lambda =?$

$X = \mathbb{F}_7$, $Q = \{1, 2, 4\}$,

$$\begin{aligned}\mathcal{B} &= \{Q + a \mid a \in \mathbb{F}_7\} \\ &= \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \ldots, \{0, 1, 3\}\}.\end{aligned}$$

- A non-complete design is a good approximation of the complete design.
- The axiom may be considered as a geometric one (projective plane). Consider the set of lines in the usual plane.

t-(v, k, λ) designs

Definition

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$: a finite set of v "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(7, 3, 1)$ design can be constructed from $Q \subset \mathbb{F}_7$.

t-(v,k,λ) designs

Definition

A t-(v,k,λ) design is a pair $(\mathcal{P},\mathcal{B})$, where

- $\mathcal{P}$: a finite set of v "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(7,3,1)$ design can be constructed from $Q \subset \mathbb{F}_7$.

t-(v, k, λ) designs

Definition

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$: a finite set of v "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(7, 3, 1)$ design can be constructed from $Q \subset \mathbb{F}_7$.

t-(v, k, λ) designs

Definition

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$: a finite set of v "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(7, 3, 1)$ design can be constructed from $Q \subset \mathbb{F}_7$.
- 2-$(q^2, q, 1)$ design $=$ affine plane of order q

t-(v, k, λ) designs

Definition

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$: a finite set of v "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(7, 3, 1)$ design can be constructed from $Q \subset \mathbb{F}_7$.
- 2-$(q^2, q, 1)$ design = affine plane of order q

t-(v,k,λ) designs

Definition

A t-(v,k,λ) design is a pair $(\mathcal{P},\mathcal{B})$, where

- $\mathcal{P}$: a finite set of v "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(7,3,1)$ design can be constructed from $Q \subset \mathbb{F}_7$.
- 2-$(q^2,q,1)$ design $=$ affine plane of order q

t-(v, k, λ) designs

Definition

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$: a finite set of v "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(7, 3, 1)$ design can be constructed from $Q \subset \mathbb{F}_7$.
- 2-$(q^2, q, 1)$ design $=$ affine plane of order q

$$t\text{-design} \implies (t-1)\text{-design}$$

Intersection numbers

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design. Write $\lambda = \lambda_t$,

$$\lambda_{t-1} = |\{B \in \mathcal{B} \mid T' \subset B\}|,$$

where $T' \subset \mathcal{P}$, $|T'| = t - 1$. Then

Intersection numbers

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design. Write $\lambda = \lambda_t$,

$$\lambda_{t-1} = |\{B \in \mathcal{B} \mid T' \subset B\}|,$$

where $T' \subset \mathcal{P}$, $|T'| = t - 1$. Then

Intersection numbers

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design. Write $\lambda = \lambda_t$,

$$\lambda_{t-1} = |\{B \in \mathcal{B} \mid T' \subset B\}|,$$

where $T' \subset \mathcal{P}$, $|T'| = t - 1$. Then

$$\begin{aligned}
\lambda_{t-1}(k - t + 1) &= \sum_{\substack{B \in \mathcal{B} \\ T' \subset B}} |B \setminus T'| \\
&= |\{(B, x) \mid B \in \mathcal{B},\ T' \cup \{x\} \subset B,\ x \in \mathcal{P} \setminus T'\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} |\{B \in \mathcal{B} \mid T' \cup \{x\} \subset B\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} \lambda_t \\
&= \lambda_t(v - t + 1).
\end{aligned}$$

Intersection numbers

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design. Write $\lambda = \lambda_t$,

$$\lambda_{t-1} = |\{B \in \mathcal{B} \mid T' \subset B\}|,$$

where $T' \subset \mathcal{P}$, $|T'| = t - 1$. Then

$$\begin{aligned}
\lambda_{t-1}(k - t + 1) &= \sum_{\substack{B \in \mathcal{B} \\ T' \subset B}} |B \setminus T'| \\
&= |\{(B, x) \mid B \in \mathcal{B},\ T' \cup \{x\} \subset B,\ x \in \mathcal{P} \setminus T'\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} |\{B \in \mathcal{B} \mid T' \cup \{x\} \subset B\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} \lambda_t \\
&= \lambda_t(v - t + 1).
\end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1} = 1 \cdot \frac{24-5+1}{8-5+1} = \frac{20}{4} = 5$$

For example,

$$5\text{-}(24, 8, 1) \implies 4\text{-}(24, 8, 5)$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1} = 5 \cdot \frac{24-4+1}{8-4+1} =$$

For example,

$$\begin{aligned} 5\text{-}(24, 8, 1) &\implies 4\text{-}(24, 8, 5) \\ &\implies 3\text{-}(24, 8, 21) \end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1} = 5 \cdot \frac{24-4+1}{8-4+1} =$$

For example,

$$\begin{aligned} 5\text{-}(24, 8, 1) &\implies 4\text{-}(24, 8, 5) \\ &\implies 3\text{-}(24, 8, 21) \end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1} = 5 \cdot \frac{24-4+1}{8-4+1} = 5 \cdot \frac{21}{5} = 21$$

For example,

$$\begin{aligned} 5\text{-}(24, 8, 1) &\implies 4\text{-}(24, 8, 5) \\ &\implies 3\text{-}(24, 8, 21) \end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1}$$

For example,

$$\begin{aligned} 5\text{-}(24, 8, 1) &\implies 4\text{-}(24, 8, 5) \\ &\implies 3\text{-}(24, 8, 21) \\ &\implies 2\text{-}(24, 8, 77) \end{aligned}$$

$(\mathcal{P},\mathcal{B})$: t-(v,k,λ) design

Then $(\mathcal{P},\mathcal{B})$: $(t-1)$-(v,k,λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1}$$

For example,

$$\begin{aligned} 5\text{-}(24,8,1) &\implies 4\text{-}(24,8,5) \\ &\implies 3\text{-}(24,8,21) \\ &\implies 2\text{-}(24,8,77) \\ &\implies 1\text{-}(24,8,253) \end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1}$$

For example,

$$\begin{aligned} 5\text{-}(24, 8, 1) &\implies 4\text{-}(24, 8, 5) \\ &\implies 3\text{-}(24, 8, 21) \\ &\implies 2\text{-}(24, 8, 77) \\ &\implies 1\text{-}(24, 8, 253) \\ &\implies 0\text{-}(24, 8, 759) \end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1}$$

For example,

$$\begin{aligned} 5\text{-}(24, 8, 1) &\implies 4\text{-}(24, 8, 5) \\ &\implies 3\text{-}(24, 8, 21) \\ &\implies 2\text{-}(24, 8, 77) \\ &\implies 1\text{-}(24, 8, 253) \\ &\implies 0\text{-}(24, 8, 759) \\ &\iff |\mathcal{B}| = 759. \end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.
Define

$$\lambda_i^j = |\{B \in \mathcal{B} \mid I \subset B,\ B \cap J = \emptyset\}|.$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.
Define

$$\lambda_i^j = |\{B \in \mathcal{B} \mid I \subset B,\ B \cap J = \emptyset\}|.$$

In particular,

$$\lambda_i^0 = \lambda_i = |\{B \in \mathcal{B} \mid I \subset B\}|.$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.
Define

$$\lambda_i^j = |\{B \in \mathcal{B} \mid I \subset B,\ B \cap J = \emptyset\}|.$$

In particular,

$$\lambda_i^0 = \lambda_i = |\{B \in \mathcal{B} \mid I \subset B\}|.$$

$$\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j.$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

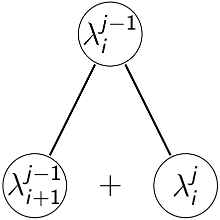
Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.
Define

$$\lambda_i^j = |\{B \in \mathcal{B} \mid I \subset B,\ B \cap J = \emptyset\}|.$$

In particular,

$$\lambda_i^0 = \lambda_i = |\{B \in \mathcal{B} \mid I \subset B\}|.$$

$$\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j.$$



$$\begin{array}{c}
\lambda_0^0 \\
\lambda_1^0\ \lambda_0^1 \\
\lambda_2^0\ \lambda_1^1\ \lambda_0^2 \\
\lambda_3^0\ \lambda_2^1\ \lambda_1^2\ \lambda_0^3 \\
\lambda_4^0\ \lambda_3^1\ \lambda_2^2\ \lambda_1^3\ \lambda_0^4 \\
\lambda_5^0\ \lambda_4^1\ \lambda_3^2\ \lambda_2^3\ \lambda_1^4\ \lambda_0^5
\end{array}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

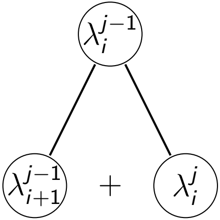
Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.
Define

$$\lambda_i^j = |\{B \in \mathcal{B} \mid I \subset B,\ B \cap J = \emptyset\}|.$$

In particular,

$$\lambda_i^0 = \lambda_i = |\{B \in \mathcal{B} \mid I \subset B\}|.$$

$$\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j.$$



$$\lambda_0^0$$
$$\lambda_1^0 \ \lambda_0^1$$
$$\lambda_2^0 \ \lambda_1^1 \ \lambda_0^2$$
$$\lambda_3^0 \ \lambda_2^1 \ \lambda_1^2 \ \lambda_0^3$$
$$\lambda_4^0 \ \lambda_3^1 \ \lambda_2^2 \ \lambda_1^3 \ \lambda_0^4$$
$$\lambda_5^0 \ \lambda_4^1 \ \lambda_3^2 \ \lambda_2^3 \ \lambda_1^4 \ \lambda_0^5$$

5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

759

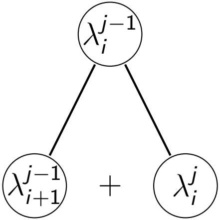
253

77

21

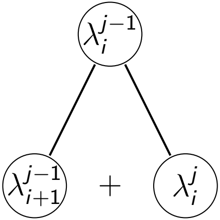
5

1



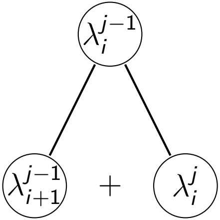
5-$(24, 8, 1)$ design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

759	
253	506
77	176
21	56
5	16
1	4



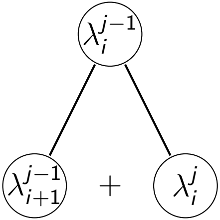
5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

					759		
				253		506	
			77		176		330
		21		56		120	
	5		16		40		
1		4		12			



5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

759
253 506
77 176 330
21 56 120 210
5 16 40 80
1 4 12 28



5-$(24, 8, 1)$ design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

759

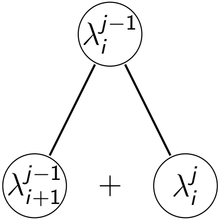
253 506

77 176 330

21 56 120 210

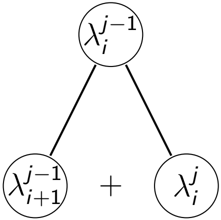
5 16 40 80 130

1 4 12 28 52



5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

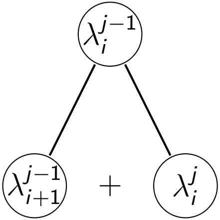
					759					
				253		506				
			77		176		330			
		21		56		120		210		
	5		16		40		80		130	
1		4		12		28		52		78



5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

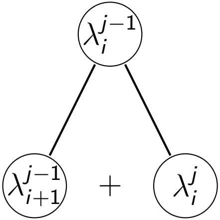
759
253 506
77 176 330
21 56 120 210
5 16 40 80 130
1 4 12 28 52 78

Next row? λ_6^0, λ_5^1, λ_4^2, ...



5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

759
253 506
77 176 330
21 56 120 210
5 16 40 80 130
1 4 12 28 52 78



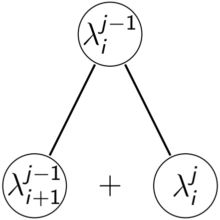
Next row? λ_6^0, λ_5^1, λ_4^2, ...

$$\lambda_6^0(I) = |\{B \in \mathcal{B} \mid I \subset B\}| = 1 \text{ or } 0$$

depending on the choice of $I \subset \mathcal{P}$ with $|I| = 6$.

5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

759
253 506
77 176 330
21 56 120 210
5 16 40 80 130
1 4 12 28 52 78



Next row? λ_6^0, λ_5^1, λ_4^2, ...

$$\lambda_6^0(I) = |\{B \in \mathcal{B} \mid I \subset B\}| = 1 \text{ or } 0$$

depending on the choice of $I \subset \mathcal{P}$ with $|I| = 6$.
Choose I in such a way that $\lambda_6^0(I) = 1$.

5-(24, 8, 1) design, $I \subset \mathcal{P}$, $|I| = 6$, $I \subset \exists B \in \mathcal{B}$

$$\lambda^j_{6-j} = |\{B \in \mathcal{B} \mid I \setminus J \subset B,\ B \cap J = \emptyset\}| \quad \text{where } J \subset I,\ J = j.$$

$$\lambda^j_{5-j} = \lambda^j_{6-j} + \lambda^{j+1}_{5-j}$$

giving

$$\begin{array}{ccccccccccccc}
 & & & & & & 759 & & & & & & \\
 & & & & & 253 & & 506 & & & & & \\
 & & & & 77 & & 176 & & 330 & & & & \\
 & & & 21 & & 56 & & 120 & & 210 & & & \\
 & & 5 & & 16 & & 40 & & 80 & & 130 & & \\
 & 1 & & 4 & & 12 & & 28 & & 52 & & 78 & \\
1 & & 0 & & 4 & & 8 & & 20 & & 32 & & 46
\end{array}$$

Similarly, taking $I \subset \mathcal{P}$, $|I| = 7$ appropriately, we obtain λ^j_{7-j}.
Finally taking $I \in \mathcal{B}$, we obtain λ^j_{8-j}.

5-(24, 8, 1) design

759								
253	506							
77	176	330						
21	56	120	210					
5	16	40	80	130				
1	4	12	28	52	78			
1	0	4	8	20	32	46		
1	0	0	4	4	16	16	30	
1	0	0	0	4	0	16	0	30

The last row implies

$$B, B' \in \mathcal{P},\ B \neq B' \implies |B \cap B'| \in \{4, 2, 0\}.$$

Binary codes

A (linear) binary code of length v is a subspace of the vector space $\mathbb{F}_2^v$. If C is a binary code and $\dim C = k$, we say C is an binary $[v, k]$ code.

Binary codes

A (linear) binary code of length v is a subspace of the vector space $\mathbb{F}_2^v$. If C is a binary code and $\dim C = k$, we say C is an binary $[v, k]$ code. The dual code of a binary code C is defined as

$$C^\perp = \{x \in \mathbb{F}_2^v \mid x \cdot y = 0 \ (\forall y \in C)\}.$$

where

$$x \cdot y = \sum_{i=1}^{v} x_i y_i.$$

Binary codes

A (linear) binary code of length v is a subspace of the vector space $\mathbb{F}_2^v$. If C is a binary code and $\dim C = k$, we say C is an binary $[v, k]$ code. The dual code of a binary code C is defined as

$$C^\perp = \{x \in \mathbb{F}_2^v \mid x \cdot y = 0 \ (\forall y \in C)\}.$$

where

$$x \cdot y = \sum_{i=1}^{v} x_i y_i.$$

Then

$$\dim C^\perp = v - \dim C.$$

The code C is said to be self-orthogonal if $C \subset C^\perp$ and self-dual if $C = C^\perp$.

Generator matrix of a code

If a binary code C is generated by row vectors $x^{(1)}, \dots, x^{(b)}$, then the matrix

$$\begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(b)} \end{bmatrix}$$

is called a generator matrix of C. This means

$$C = \{\sum_{i=1}^{b} \epsilon_i x^{(i)} \mid \epsilon_1, \dots, \epsilon_b \in \mathbb{F}_2\} \subset \mathbb{F}_2^v.$$

Generator matrix of a code

If a binary code C is generated by row vectors $x^{(1)}, \ldots, x^{(b)}$, then the matrix

$$\begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(b)} \end{bmatrix}$$

is called a generator matrix of C. This means

$$C = \{\sum_{i=1}^{b} \epsilon_i x^{(i)} \mid \epsilon_1, \ldots, \epsilon_b \in \mathbb{F}_2\} \subset \mathbb{F}_2^v.$$

Note

$$C \subset C^\perp \iff |\operatorname{supp}(x^{(i)}) \cap \operatorname{supp}(x^{(j)})| \equiv 0 \pmod 2 \quad (\forall i, j).$$

Incidence matrix of a design

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design, the incidence matrix $M(\mathcal{D})$ of $\mathcal{D}$ is the $|\mathcal{B}| \times |\mathcal{P}|$ matrix whose rows and columns are indexed by $\mathcal{B}$ and $\mathcal{P}$, respectively, such that

$$(M(\mathcal{D}))_{B,p} = \begin{cases} 1 & \text{if } p \in B, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the row vectors of $M(\mathcal{D})$ are the characteristic vectors of blocks.

Incidence matrix of a design

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design, the incidence matrix $M(\mathcal{D})$ of $\mathcal{D}$ is the $|\mathcal{B}| \times |\mathcal{P}|$ matrix whose rows and columns are indexed by $\mathcal{B}$ and $\mathcal{P}$, respectively, such that

$$(M(\mathcal{D}))_{B,p} = \begin{cases} 1 & \text{if } p \in B, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the row vectors of $M(\mathcal{D})$ are the characteristic vectors of blocks.

The binary code of the design $\mathcal{D}$ is the binary code of length v having $M(\mathcal{D})$ as a generator matrix.

Incidence matrix of a design

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design, the incidence matrix $M(\mathcal{D})$ of $\mathcal{D}$ is the $|\mathcal{B}| \times |\mathcal{P}|$ matrix whose rows and columns are indexed by $\mathcal{B}$ and $\mathcal{P}$, respectively, such that

$$(M(\mathcal{D}))_{B,p} = \begin{cases} 1 & \text{if } p \in B, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the row vectors of $M(\mathcal{D})$ are the characteristic vectors of blocks.

The binary code of the design $\mathcal{D}$ is the binary code of length v having $M(\mathcal{D})$ as a generator matrix.

$\dim C \le 12$ for 5-$(24, 8, 1)$ design

Recall that in a 5-$(24, 8, 1)$ design $(\mathcal{P}, \mathcal{B})$,

$$|B \cap B'| \in \{8, 4, 2, 0\} \quad (\forall B, B' \in \mathcal{B}).$$

dim $C \leq 12$ for 5-$(24, 8, 1)$ design

Recall that in a 5-$(24, 8, 1)$ design $(\mathcal{P}, \mathcal{B})$,

$$|B \cap B'| \in \{8, 4, 2, 0\} \quad (\forall B, B' \in \mathcal{B}).$$

The binary code C of a 5-$(24, 8, 1)$ design is self-orthogonal. Indeed, the incidence matrix has row vectors $x^{(B)}$ $(B \in \mathcal{B})$, the characteristic vector of the block B. Then

$$x^{(B)} \cdot x^{(B')} = |B \cap B'| \bmod 2 = (8 \text{ or } 4 \text{ or } 2 \text{ or } 0) \bmod 2 = 0.$$

dim $C \leq 12$ for 5-$(24, 8, 1)$ design

Recall that in a 5-$(24, 8, 1)$ design $(\mathcal{P}, \mathcal{B})$,

$$|B \cap B'| \in \{8, 4, 2, 0\} \quad (\forall B, B' \in \mathcal{B}).$$

The binary code C of a 5-$(24, 8, 1)$ design is self-orthogonal. Indeed, the incidence matrix has row vectors $x^{(B)}$ $(B \in \mathcal{B})$, the characteristic vector of the block B. Then

$$x^{(B)} \cdot x^{(B')} = |B \cap B'| \bmod 2 = (8 \text{ or } 4 \text{ or } 2 \text{ or } 0) \bmod 2 = 0.$$

Thus $C \subset C^{\perp}$, hence

$$\dim C \leq \frac{1}{2}(\dim C + \dim C^{\perp}) \leq \frac{24}{2} = 12.$$

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1 1 2 3 4 5 6 7 8

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8	
B_2	1	2	3	4	5	6	7	8	9

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1 1 2 3 4 5 6 7 8
B_2 1 2 3 4 5 6 7 8 9 10 11 12

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8				
B_2	1	2	3	4					9	10	11	12

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8				
B_2	1	2	3	4					9	10	11	12
B_3	1	2	3		5				9			

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8				
B_2	1	2	3	4					9	10	11	12
B_3	1	2	3	4	5	6	7	8	9	10	11	12

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8							
B_2	1	2	3	4					9	10	11	12			
B_3	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8							
B_2	1	2	3	4					9	10	11	12			
B_3	1	2	3		5				9				13	14	15

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8							
B_2	1	2	3	4					9	10	11	12			
B_3	1	2	3		5				9				13	14	15
B_4	1	2		4	5				9						

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8							
B_2	1	2	3	4					9	10	11	12			
B_3	1	2	3		5				9				13	14	15
B_4	1	2		4	5				9						

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8							
B_2	1	2	3	4					9	10	11	12			
B_3	1	2	3		5				9				13	14	15
B_4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8										
B_2	1	2	3	4					9	10	11	12						
B_3	1	2	3		5				9				13	14	15			
B_4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8										
B_2	1	2	3	4					9	10	11	12						
B_3	1	2	3		5				9				13	14	15			
B_4	1	2		4	5				9							16	17	18

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8									
B_2	1	2	3	4					9	10	11	12					
B_3	1	2	3		5				9				13	14	15		
B_4	1	2		4	5				9							16	17	18
B_5	1		3	4	5				9									

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8													
B_2	1	2	3	4					9	10	11	12									
B_3	1	2	3		5				9				13	14	15						
B_4	1	2		4	5				9							16	17	18			
B_5	1		3	4	5				9										19	20	21

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8													
B_2	1	2	3	4					9	10	11	12									
B_3	1	2	3		5				9				13	14	15						
B_4	1	2		4	5				9							16	17	18			
B_5	1		3	4	5				9										19	20	21
B_6		2	3	4	5				9												

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9															

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15									

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15									

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18						

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21			

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9															

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9															

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2	3	4	5	6	7	8	9	10	11	12				16	17	18	19			22		

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2	3	4	5	6	7	8	9	10	11	12				16	17	18	19			22		

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19			22		

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19			22		

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22		

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22		

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9															

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9															

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1	2	3	4	2	6	7	8	9	10	11	12	13			16			19	20	21	22	23	

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1	2	3	4	2	6	7	8	9	10	11	12	13			16			19	20	21	22	23	

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1	2	3	4	2	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9															

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2	3	4	5	6	7	8	9				13	14	15	16	17	18	19	20		22	23	

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9															

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1	2	3	4	5	6	7	8	9	10			13	14	15	16	17		19	20	21	22		24

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3						9								17							

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3						9								17							

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3	4	5	6			9	10	11	12	13	14	15	16	17	18	19			22		24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3	4	5	6			9	10	11	12	13	14	15	16	17	18	19			22		24

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3	4	5	6			9	10	11	12	13	14	15	16	17	18	19		21	22	23	24

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3	4	5	6			9	10	11	12	13	14	15	16	17	18	19		21	22	23	24

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3	4	5	6			9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3	4	5	6			9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3				7		9								17				21		23	

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8																
B_2	1	2	3	4					9	10	11	12												
B_3	1	2	3		5				9				13	14	15									
B_4	1	2		4	5				9							16	17	18						
B_5	1		3	4	5				9										19	20	21			
B_6		2	3	4	5				9													22	23	24
B_7	1	2	3			6			9							16			19			22		
B_8	1	2		4		6			9				13							20			23	
B_9	1		3	4		6			9					14			17							24
B_{10}	1	2			5	6			9	10											21			24
B_{11}	1		3		5	6			9		11							18					23	
B_{12}	1	2	3				7		9								17				21		23	

The characteristic vectors of these 12 blocks generate a unique 12-dimensional code called the Golay code.

Summary

$\mathcal{D}$: 5-$(24, 8, 1)$ design.

- The binary code C of $\mathcal{D}$ is a self-dual $[24, 12]$ code.

Summary

$\mathcal{D}$: 5-$(24, 8, 1)$ design.

- The binary code C of $\mathcal{D}$ is a self-dual $[24, 12]$ code.
- There is a unique 5-$(24, 8, 1)$ design up to isomorphism. This was proved by E. Witt (1938).

Summary

$\mathcal{D}$: 5-$(24, 8, 1)$ design.

- The binary code C of $\mathcal{D}$ is a self-dual $[24, 12]$ code.
- There is a unique 5-$(24, 8, 1)$ design up to isomorphism. This was proved by E. Witt (1938).
- There is a way to construct the code C directly, using quadratic residues $Q \subset \mathbb{F}_{23}$.

Summary

$\mathcal{D}$: 5-$(24, 8, 1)$ design.

- The binary code C of $\mathcal{D}$ is a self-dual $[24, 12]$ code.
- There is a unique 5-$(24, 8, 1)$ design up to isomorphism. This was proved by E. Witt (1938).
- There is a way to construct the code C directly, using quadratic residues $Q \subset \mathbb{F}_{23}$.

Let D be the matrix of quadratic residue characters for $\mathbb{F}_{23}$. Then C is the code generated by the row vectors of the matrix

$$G = \begin{bmatrix} & 1 \\ \frac{1}{2}(J - I - D) & \vdots \\ & 1 \end{bmatrix} \quad (23 \times 24 \text{ matrix}).$$