A variation of Godsil–McKay switching

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joint work with Ferdinand Ihringer

November 22, 2018
International Workshop on Algebraic Combinatorics
Anhui University

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Wang–Xu (2006): "=" $\iff$ determined by "generalized" spectrum.

Godsil–McKay switching

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3. Abiad, Haemers (2016), Kubota (2016): symplectic graphs

4. Barwick, Jackson, Penttila (2017): orthogonal polar
5. Ihringer (2017): all polar
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Godsil–McKay switching

$\Gamma = (X, E)$: graph, $X = (\bigcup_i C_i) \cup D$.
Assume $\forall x \in D$, $\forall i$, x is adjacent to $0, 1/2$ or all vertices of C_i.

Godsil–McKay switching: interchange adj. and non-adj. between $x \in D$ and C_i if x is adj. to $1/2$ of C_i.

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Theorem (Godsil–McKay, 1982)

If $\{C_i\}_i$ is equitable, then the resulting graph is cospectral with the original.

Equitable: $\forall i$, $\forall x, \forall y \in C_i$, $\forall j$, $|\Gamma(x) \cap C_j| = |\Gamma(y) \cap C_j|$.

Godsil–McKay switching with one cell C

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In this special case:

Theorem (Godsil–McKay, 1982)

If the subgraph of Γ induced on C is regular, then the resulting graph is cospectral with the original.

One cell of size 4

$\Gamma = (X, E)$: graph, $X = C \cup D$, $|C| = 4$.
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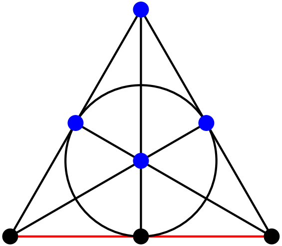
If $|C| = 2$, then the switched graph is isomorphic to the original.

Fano plane in a polar space

A quadrangle C in a Fano plane.



Every line meets C at 0 or 2 points.

$\Uparrow$

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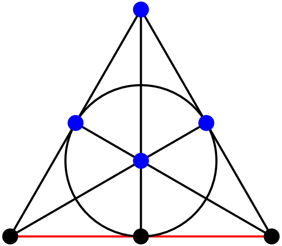

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Neighbors of a vertex outside C

$\Longrightarrow$ C can be used in Godsil–McKay switching.

Polar space

Let V be a vector space over $\mathbb{F}_q$ with nondegenerate

$$\left\{ \begin{array}{l} \text{symplectic} \\ \text{hermitian} \\ \text{symmetric bilinear} \end{array} \right\} \text{ form } B.$$

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Strongly regular polar graph Γ: $\mathbb{P}$ as vertices,

$$x \sim y \iff x \subseteq y^\perp.$$

That is,

$$\Gamma(x) = x^\perp \cap \mathbb{P}.$$

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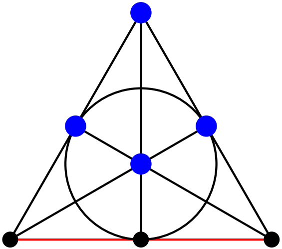

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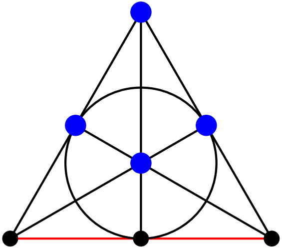

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If this plane P is totally isotropic, then

$$\Gamma(x) \cap P = x^{\perp} \cap P = \text{a line of } P \text{ or } P$$
$$\implies |\Gamma(x) \cap C| = 0, 2 \text{ or } 4$$

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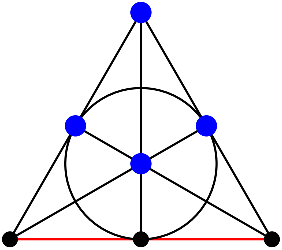

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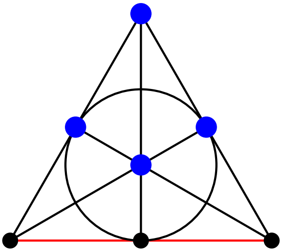

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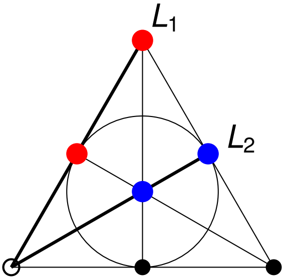
$\Longrightarrow$ C can be used in Godsil–McKay switching.

One cell of size 4 partitioned into 2 parts

$C = C_1 \cup C_2 \qquad C_i = L_i \setminus (L_1 \cap L_2).$

A quadrangle is a union of two lines minus the point of intersection.



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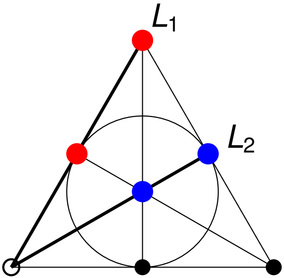

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$$\Gamma(x) \cap C = C_1 \text{ or } C_2 \text{ or one point each from } C_i \text{, or } C$$

Theorem

Let Γ be a graph whose vertex set is partitioned as $C_1 \cup C_2 \cup D$, where $|C_1| = |C_2| = 2$. Assume that the subgraph of Γ induced on C is regular, and that

$$|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|, \text{ or}$$
$$\Gamma(x) \cap (C_1 \cup C_2) = C_1 \text{ or } C_2.$$

Construct a graph $\bar{\Gamma}$ from Γ by modifying edges between C and D as follows:

$$\bar{\Gamma}(x) \cap C = \begin{cases} C_2 & \text{if } \Gamma(x) \cap C = C_1, \\ C_1 & \text{if } \Gamma(x) \cap C = C_2, \\ \Gamma(x) \cap C & \text{otherwise,} \end{cases}$$

for $x \in D$. Then $\bar{\Gamma}$ is cospectral with Γ.

Proof: $A(\bar{\Gamma}) = P^\top A(\Gamma) P$

$$A(\Gamma) = \begin{array}{c} \\ C_1 \\ C_2 \\ D \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & D \end{array} \\ \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline * & * & * \end{array}\right] \end{array}$$

$$P = \begin{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} & 0 \\ 0 & I_D \end{bmatrix} \in O(n, \mathbb{Q}).$$

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$$Q = \begin{bmatrix} \frac{1}{2}(J - 2I) & 0 \\ 0 & I_D \end{bmatrix},$$

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but $PQ^\top$ is a permutation matrix, resulting in:

$$P^\top A(\Gamma) P \text{ isomorphic } Q^\top A(\Gamma) Q.$$

Projective space of order $q > 2$

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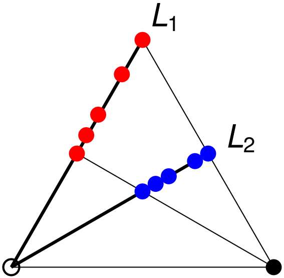
Union of two lines minus the point of intersection. $|C| = 2q$.



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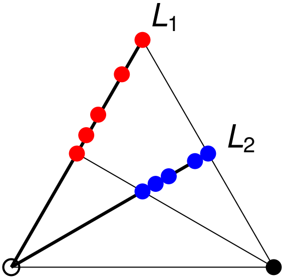

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□

The original Godsil–McKay switching (with one cell C) uses

$$Q = \begin{bmatrix} \frac{1}{q}J - I & \frac{1}{q}J & 0 \\ \frac{1}{q}J & \frac{1}{q}J - I & 0 \\ 0 & 0 & I_D \end{bmatrix} = \begin{bmatrix} \frac{1}{q}J - I & 0 \\ 0 & I_D \end{bmatrix}.$$

$$\left[*\right] \left[\frac{1}{q}J - I\right] = \begin{cases} \mathbf{1} - * & \text{if } *J = q\mathbf{1} \quad (|\Gamma(x) \cap C| = \frac{1}{2}|C|) \\ \left[*\right] & \text{if } \left[*\right] = 0 \text{ or } \mathbf{1} \end{cases}$$

Hypotheses of the two switchings

Two switchings require different hypotheses.

Godsil–McKay: for $|C| = 2q$,

$$|\Gamma(x) \cap C| = 0, q \text{ or } 2q$$

Ours: for $C = C_1 \cup C_2$, $|C_1| = |C_2| = q$,

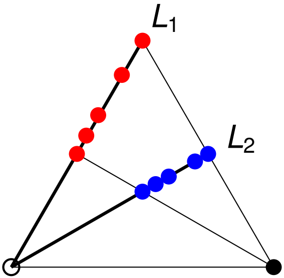
$$|\Gamma(x) \cap C| \text{ could possibly be any even number}$$

For $q > 2$, these two methods in general give non-isomorphic graphs.

Question: Is there a common generalization?

Projective space of order $q > 2$

$C = C_1 \cup C_2 \qquad C_i = L_i \setminus (L_1 \cap L_2)$.
Union of two lines minus the point of intersection. $|C| = 2q$.



For any line L,
$|L \cap C_1| = |L \cap C_2|$, or
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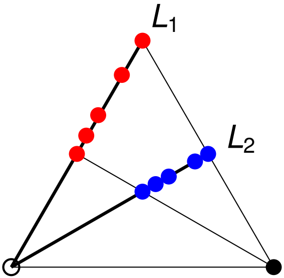

For any line L,
$|L \cap C_1| = |L \cap C_2| = 1$, or
$L \cap (C_1 \cup C_2) = C_1$ or C_2.

Let Γ be the graph of non-isotropic points in a hermitian polar space. Two vertices are adjacent iff orthogonal. If C consists entirely of non-isotropic points, the switching can be applied.

Non-isotropic points

Let V be a vector space over $\mathbb{F}_{q^2}$ equipped with a nondegenerate hermitian form.

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Then Γ is a strongly regular graph.

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For all cliques $\{x, y, z\}$ of Γ, $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)|$ is independent of the choice of $\{x, y, z\}$.

After switching, this property will be violated
$\Longrightarrow$ the resulting cospectral graph is not isomorphic to the original graph Γ.

Switching Γ to $\bar{\Gamma}$

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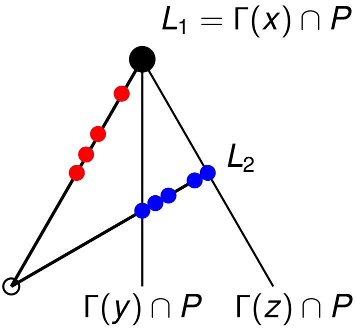

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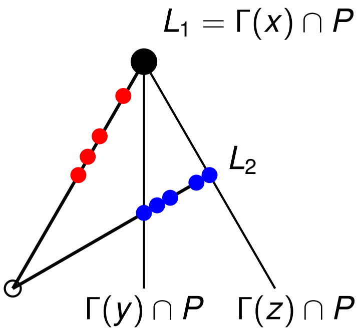

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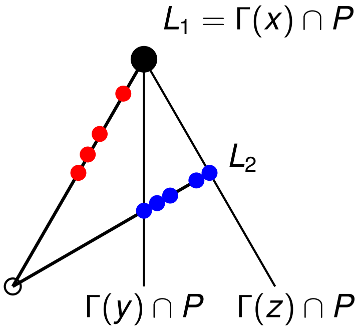

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$$|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z) \cap P| > |\bar{\Gamma}(x) \cap \bar{\Gamma}(y) \cap \bar{\Gamma}(z) \cap P|.$$

Therefore, $\Gamma \not\cong \bar{\Gamma}$.

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Thank you very much for your attention!