Krein parameters of fiber-commutative coherent configurations

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Krein condition for coherent configurations

S. A. Hobart, Linear Algebra Appl. 226/228 (1995), 499–508.

> *In our applications . . . , we use $Z = Z' = \phi_s(J)$, where J is the all 1s matrix. Other choices do not produce any new results for these particular examples.*

The goal of this talk is to clarify this claim by proving it in a more general setting (fiber-commutative).

In doing so, we develop a theory analogous to commutative coherent configurations = association schemes

History

L. L. Scott (1973) attributes the discovery of the source of Krein condition

$$q_{ij}^k \geq 0$$

to C. Dunkl, who attributes the condition itself to the work of M. G. Krein (1950). P. Delsarte (1973) formulated and proved the inequality for association schemes.

The indices i, j, k range over a set of irreducible representations appearing in a particular module in question.

The parameters q_{ij}^k are called Krein parameters.

A special case is the tensor product coefficients for irreducible characters of finite groups.

Cameron, Goethals and Seidel (1978) related Krein parameters to Norton algebras.

Combinatorial applications

Properties of Krein parameters:

- Krein conditions
- Absolute bounds

are used to rule out existence of certain putative strongly regular graphs.

See Brouwer's database of strongly regular graphs.

Coherent configuration = coherent algebra

A $\mathbb{C}$-subspace $\mathcal{A} \subset M_n(\mathbb{C})$ is called a coherent algebra if

- closed under matrix product,
- $I \in \mathcal{A}$,
- closed under entrywise product,
- $J \in \mathcal{A}$,
- closed under conjugate-transpose $*$.

$\implies \exists\{A_i \mid i \in \Lambda\}$: basis of $\mathcal{A}$, $(0,1)$-matrices, with

$$\sum_{i \in \Lambda} A_i = J, \quad \{A_i \mid i \in \Lambda\} = \{A_i^\top \mid i \in \Lambda\}.$$

The trivial coherent algebra: $\langle I, J \rangle$, $M_n(\mathbb{C})$.

Strongly regular graph

Let A be the adjacency matrix of an undirected graph G. Then the 3-dimensional vector space

$$\mathcal{A} = \langle I, A, J - I - A \rangle$$

is a (commutative) coherent algebra if and only if G is a strongly regular graph, i.e.,

$$\begin{aligned} AJ &= kJ, \\ A^2 &= kI + \lambda A + \mu(J - I - A) \end{aligned}$$

for some k, λ, μ.

Projective plane $(\mathcal{P}, \mathcal{L})$

It is an incidence structure consists of points $\mathcal{P}$, lines $\mathcal{L}$ with incidence relation between them, satisfying certain axioms. It can be described by a set of matrices whose rows and columns are indexed by $\mathcal{P} \cup \mathcal{L}$:

$$\begin{array}{cc} & \begin{array}{cc} \mathcal{P} & \mathcal{L} \end{array} \\ \begin{array}{c} \mathcal{P} \\ \mathcal{L} \end{array} & \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{array}$$

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} J-I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & J-I \end{pmatrix},$$
$$\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J-M \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M^\top & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ J-M^\top & 0 \end{pmatrix}$$

Commutative coherent algebra = association scheme

$$M_n(\mathbb{C}) \supset \mathcal{A} = \langle A_i \mid i \in \Lambda \rangle = \bigoplus_i \mathbb{C} E_i.$$

$$E_i E_j = \delta_{ij} E_i.$$

$$E_i \circ E_j = \frac{1}{n} \sum_k q_{ij}^k E_k.$$

The scalars q_{ij}^k are called Krein parameters. Krein condition asserts $q_{ij}^k \geq 0$. To see this, it suffices to invoke

Lemma

Let $A, B \in M_n(\mathbb{C})$ be Hermitian matrices. If $A, B \succeq 0$, then $A \circ B \succeq 0$.

Proof.

$A \otimes B \succeq 0$ and it contains $A \circ B$ as a principal submatrix. □

Krein condition

We could begin with a commutative algebra

$$\mathcal{A} = \langle A_i \mid i \in \Lambda \rangle$$

defined by structure constants:

$$A_i A_j = \sum_k p_{ij}^k A_k.$$

With modest hypothesis, it has decomposition

$$\mathcal{A} = \bigoplus_i \mathbb{C} E_i, \quad E_i E_j = \delta_{ij} E_i.$$

Define $\circ$ by $A_i \circ A_j = \delta_{ij} A_i$ (and extend by linearity). Define q_{ij}^k by

$$E_i \circ E_j = \sum_k q_{ij}^k E_k.$$

If $q_{ij}^k \geq 0$ fails, then $\mathcal{A}$ cannot be a coherent algebra (there cannot be a coherent algebra with structure constants p_{ij}^k).

Non-commutative case

Let $\mathcal{A}$ be a (not necessarily commutative) coherent algebra.

$$M_n(\mathbb{C}) \supset \mathcal{A} = \bigoplus_i \mathcal{I}_i,$$
$$\mathcal{I}_i \cong M_{e_i}(\mathbb{C}) = \mathbb{C} \quad \text{($*$-isomorphic)}$$
$$\mathcal{I}_i = \mathcal{A}E_i\mathcal{A} = \mathcal{A}E_i = \mathbb{C}E_i$$

Let $\mathcal{P}(\cdot)$ denote the subset of Hermitian positive semidefinite matrices:

$$\mathcal{P}(\cdot) = \{Z \in \cdot \mid Z \succeq 0\}.$$

Krein condition (for coherent configurations) asserts

$$\forall F \in \mathcal{P}(\mathcal{I}_i),\ \forall F' \in \mathcal{P}(\mathcal{I}_j),\ F \circ F' \succeq 0 \quad E_i \circ E_j \succeq 0$$

or equivalently $(F \circ F')E_k \in \mathcal{P}(\mathcal{I}_k)$ for all k.

Summary of results

commutative	fiber-commutative
(central) primitive idempotents	basis of matrix units
Krein parameters q_{ij}^k	matrix of Krein parameters Q_{ij}^k essentially unique
Krein condition $q_{ij}^k \geq 0$	Krein condition $Q_{ij}^k \succeq 0$
absolute bound $\sum_{q_{ij}^k \neq 0} m_k \leq m_i m_j$	absolute bound $\sum_k m_k \operatorname{rank} Q_{ij}^k \leq m_i m_j$

$\mathcal{A} = \bigoplus \mathcal{A}_{ij} = \bigoplus \mathcal{I}_k$

Recall, for a projective plane,

$$\begin{array}{c} \\ \mathcal{P} \\ \mathcal{L} \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{P} & \mathcal{L} \end{array} \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{array}.$$

In general,

$$\mathcal{A} = \left(\begin{array}{c|c|c} \mathcal{A}_{11} & \mathcal{A}_{12} & * \\ \hline \mathcal{A}_{21} & \mathcal{A}_{22} & * \\ \hline * & * & * \end{array}\right) = \bigoplus_{i,j} \mathcal{A}_{ij} = \bigoplus_k \mathcal{I}_k, \quad \mathcal{I}_k \cong M_{e_k}(\mathbb{C}).$$

We say $\mathcal{A}$ is fiber-commutative if $\mathcal{A}_{ii}$ is commutative for all i.

Lemma (Hobart–Williford, 2014)

If $\mathcal{A}$ is fiber-commutative, then $\dim \mathcal{A}_{ij} \cap \mathcal{I}_k = 0$ or 1 for all i, j, k.

$\mathcal{A} = \bigoplus \mathcal{A}_{ij} = \bigoplus \mathcal{I}_k$

To avoid cumbersome notation, we fix $\mathcal{I} = \mathcal{I}_{k_0}$. Let E be the corresponding central idempotent:

$$\mathcal{I} = \mathcal{A}E\mathcal{A} = \mathcal{A}E.$$

Since $\mathcal{I} \cong M_e(\mathbb{C})$ ($*$-isomorphic) for some e, $\mathcal{I}$ has a basis of matrix units $\{e_{ij}\}$:

$$e_{ij}e_{k\ell} = \delta_{jk}e_{il}.$$

Then

$$\mathcal{P}(\mathcal{I}) = \{\sum_{i,j} z_{ij}e_{ij} \mid (z_{ij}) \in \mathcal{P}(M_e(\mathbb{C}))\}.$$

Krein condition asserts (in particular)

$$\forall F, F' \in \mathcal{P}(\mathcal{I}),\ (F \circ F')E \in \mathcal{P}(\mathcal{I}).$$

$\mathcal{A} = \bigoplus \mathcal{A}_{ij}, \mathcal{I} = \langle e_{ij} \mid 1 \le i, j \le e \rangle$

Lemma (Hobart–Williford, 2014)

If $\mathcal{A}$ is fiber-commutative, then $\dim \mathcal{A}_{ij} \cap \mathcal{I} = 0$ or 1 for all i, j.

Since

$$e_{ij} e_{k\ell} = \delta_{jk} e_{i\ell},$$
$$\mathcal{A}_{ij} \mathcal{A}_{k\ell} \subset \delta_{jk} \mathcal{A}_{i\ell},$$

we may assume without loss of generality $e_{ij} \in \mathcal{A}_{ij}$. So,

$$\bigoplus_{i,j} \mathcal{A}_{ij} = \begin{array}{|c|c|c|} \hline * & * & * \\ \hline * & * & * \\ \hline * & * & * \\ \hline \end{array} \supset \mathcal{I} = \begin{array}{|c|c|c|} \hline e_{11} & e_{12} & 0 \\ \hline e_{21} & e_{22} & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

$\mathcal{P}(\mathcal{I}) = \{\sum_{i,j} z_{ij} e_{ij} \mid (z_{ij}) \in \mathcal{P}(M_e(\mathbb{C}))\}$

For $F = \sum z_{ij} e_{ij}$, $F' = \sum z'_{ij} e_{ij} \in \mathcal{P}(\mathcal{I})$, Krein condition asserts

$$(F \circ F')E \succeq 0.$$

Since $e_{ij} \in \mathcal{A}_{ij}$ and $\mathcal{A}_{ij} \circ \mathcal{A}_{k\ell} = 0$ if $(i,j) \neq (k,\ell)$,

$$e_{ij} \circ e_{k\ell} = 0 \quad \text{if } (i,j) \neq (k,\ell).$$

Since $\mathcal{A}_{ij} E = E \mathcal{A}_{ij} \subseteq \mathcal{A}_{ij} \cap \mathcal{I} = \mathbb{C} e_{ij}$,

$$(e_{ij} \circ e_{ij})E = q_{ij} e_{ij} \quad \text{for some } q_{ij} \in \mathbb{C}.$$

Thus

$$\begin{aligned}(F \circ F')E &= \left(\left(\sum z_{ij} e_{ij}\right) \circ \left(\sum z'_{ij} e_{ij}\right)\right) E \\ &= \sum z_{ij} z'_{ij} q_{ij} e_{ij} \\ &= \sum (Z \circ Z' \circ Q)_{ij} e_{ij}\end{aligned}$$

where $Z = (z_{ij})$, $Z' = (z'_{ij})$, $Q = (q_{ij})$.

$\mathcal{P}(\mathcal{I}) = \{\sum_{i,j} z_{ij} e_{ij} \mid (z_{ij}) \in \mathcal{P}(M_e(\mathbb{C}))\}$

Recall $Q = (q_{ij})$ is defined by $(e_{ij} \circ e_{ij})E = q_{ij} e_{ij}$.

$$
\begin{aligned}
&(F \circ F')E \succeq 0 \quad (\forall F, F' \in \mathcal{P}(\mathcal{I})) \\
&\iff Z \circ Z' \circ Q \succeq 0 \quad (\forall Z, Z' \in \mathcal{P}(M_e(\mathbb{C}))) \\
&\iff Q \succeq 0.
\end{aligned}
$$

Note $J \circ J \circ Q = Q$. This explains Hobart's observation:

> *In our applications ..., we use $Z = Z' = \phi_s(J)$, where J is the all 1s matrix. Other choices do not produce any new results for these particular examples.*

Linear Algebra Appl. 226/228 (1995), p. 502.

Theorem

For a fiber-commutative coherent algebra $\mathcal{A} = \bigoplus_k \mathcal{I}_k$, *where* $\mathcal{I}_k = \mathcal{A}E_k \cong M_{e_k}(\mathbb{C}) = \langle e^k_{ij} \mid 1 \le i, j \le e_k \rangle$, *Krein condition*

$$(F \circ F')E_k \succeq 0 \quad (\forall F \in \mathcal{P}(\mathcal{I}_i),\ \forall F' \in \mathcal{P}(\mathcal{I}_j))$$

is equivalent to

$$Q^k_{ij} \succeq 0,$$

where Q^k_{ij} *is the "matrix of Krein parameters" defined by*

$$e^i_{\ell m} \circ e^j_{\ell m} = \frac{1}{\text{scalar}} \sum_k (Q^k_{ij})_{\ell m} e^k_{\ell m}.$$

Moreover, Q^k_{ij} *is essentially unique.*

Q_{ij}^k is essentially unique

Indeed, a basis of matrix units $\{e_{ij}^k \mid 1 \le i, j \le e_k\}$ for $\mathcal{I}_k \cong M_{e_k}(\mathbb{C})$ is essentially unique, since

$$\dim \mathcal{A}_{ij} \cap \mathcal{I}_k = 0 \text{ or } 1.$$

Uniqueness is up to scalar multiplication by a complex number of absolute value 1.
This results in the uniqueness of Q_{ij}^k up to entrywise multiplication by a rank-one hermitian matrix:

$$\begin{pmatrix} a & \overline{b} \\ b & c \end{pmatrix} \sim \begin{pmatrix} a & \overline{b\zeta} \\ b\zeta & c \end{pmatrix} = \begin{pmatrix} a & \overline{b} \\ b & c \end{pmatrix} \circ \left(\begin{pmatrix} 1 \\ \zeta \end{pmatrix} \begin{pmatrix} 1 & \overline{\zeta} \end{pmatrix} \right).$$

Thank you very much for your attention!