Outline

1. Grassmann graphs
2. Twisted Grassmann graphs
3. Godsil-McKay switching
4. Wang-Qiu-Hu switching
5. Strongly regular graph from polar spaces

Grassmann graph $J_q(n,k)$.

- V: n-dim. vector space over $\mathrm{GF}(q)$
- Projective geometry $\bigcup_{j=0}^{n} \begin{bmatrix} V \\ j \end{bmatrix}$
- Vertices: $\begin{bmatrix} V \\ k \end{bmatrix}$
- $W_1 \sim W_2 \iff \dim W_1 \cap W_2 = k-1$.

For the twisted Grassmann graph, take

$$n = 2d+1, \quad k = d+1.$$

In the Grassmann graph $J_q(2d+1, d+1)$:

- V: $(2d+1)$-dim. vector space over $\mathrm{GF}(q)$
- Vertices: $\begin{bmatrix} V \\ d+1 \end{bmatrix}$
- $W_1 \sim W_2 \iff \dim W_1 \cap W_2 = d$.

Fix $H \in \begin{bmatrix} V \\ 2d \end{bmatrix}$. Then $\begin{bmatrix} V \\ d+1 \end{bmatrix} = C \cup D$, where

$$C = \begin{bmatrix} V \\ d+1 \end{bmatrix} \setminus \begin{bmatrix} H \\ d+1 \end{bmatrix},$$
$$D = \begin{bmatrix} H \\ d+1 \end{bmatrix}.$$

Then

$$C = \begin{bmatrix} V \\ d+1 \end{bmatrix} \setminus \begin{bmatrix} H \\ d+1 \end{bmatrix} = \bigcup_{U \in \left[\substack{H \\ d}\right]} C_U,$$

where

$$C_U = \{W \in \begin{bmatrix} V \\ d+1 \end{bmatrix} \mid W \cap H = U\}.$$

Observe, for $W \in D$,

$$\begin{aligned} &\exists W' \in C_U,\ W \sim W' \\ &\iff W \supseteq U \\ &\iff \forall W' \in C_U,\ W \sim W'. \end{aligned}$$

Fix a polarity $\perp$ of H. Redefine edges between C and D by changing

$$W \sim W' \quad (\forall W' \in C_U \text{ with } U \subseteq W)$$

to

$$W \sim W' \quad (\forall W' \in C_{U^\perp} \text{ with } U \subseteq W)$$

for $W \in D$.
In general, $U \neq U^\perp$. In this case $C_U \cap C_{U^\perp} = \emptyset$.
The resulting graph is the twisted Grassmann graph $\tilde{J}_q(2d+1, d+1)$.

For $W \in D$, W is adjacent to

$$\begin{aligned} &\text{all of } C_U \cup C_{U^\perp} && \text{if } W \supseteq U + U^\perp \\ &\text{all of } C_U && \text{if } W \supseteq U \\ &\text{none of } C_{U^\perp} && \text{if } W \not\supseteq U^\perp \\ &\text{none of } C_U \cup C_{U^\perp} && \text{if } W \not\supseteq U, W \not\supseteq U^\perp \end{aligned}$$

"All," "Half," or "None."
The operation is to switch adjacency when "Half" takes place.

Definition of Godsil-McKay switching (1982).
$V(\Gamma) = C \cup D$, $C = \bigcup_i C_i$. Assume

$$\forall i,\ \forall x \in D,\ |\Gamma(x) \cap C_i| = |C_i|, \frac{1}{2}|C_i|, 0$$

Build a new graph $\tilde{\Gamma}$ by redefine $\Gamma(x)$ by

$$\tilde{\Gamma}(x) = C_i \setminus \Gamma(x),$$

provided

$$|\Gamma(x) \cap C_i| = \frac{1}{2}|C_i|.$$

Theorem (Godsil–McKay, 1982)

If $\{C_i\}_i$ is *equitable*, then the resulting graph is cospectral with the original.

Equitable: $\forall i$, $\forall x \in C_i, \forall y \in C_i$, $\forall j$, $|\Gamma(x) \cap C_j| = |\Gamma(y) \cap C_j|$.

2005 twisted Grassmann graphs of Van Dam–Koolen

2009 distorted geometric design, Jungnickel–Tonchev

2011 equivalence of these two, M.–Tonchev

2017 distorted$\leftrightarrow$Godsil–McKay switching

$$\begin{array}{ccc} \mathrm{PG}_d(2d,q) & \xrightarrow{\text{block graph}} & J_q(2d+1,d+1) \\ {\scriptstyle\text{distort}}\downarrow & & \downarrow{\scriptstyle\text{GM switching}} \\ \text{new design} & \xrightarrow{\text{block graph}} & \tilde{J}_q(2d+1,d+1) \end{array}$$

Block graph = graph with blocks as vertices, adjacent iff intersect at maximal size.

- The original definition of $\tilde{J}_q(2d+1,d+1)$ does not use a polarity.
- Both distorting and GM switching rely on a polarity.

Why cospectral?

A = adjacency matrix of Γ
A' = adjacency matrix of the switched graph

Then there exists $Q \in O_{|V(\Gamma)|}(\mathbb{Q})$ such that $Q^{-1}AQ = A'$.
To describe Q explicitly, consider the case $C = C_1$, and $\forall x \in D$ is adjacent to all, half, or none of C.

Let $m = |C|$, and consider the trivial coherent algebra $\langle I, J \rangle \subseteq M_m(\mathbb{C})$. It has primitive idempotents

$$E_0 = \frac{1}{m}J, \quad E_1 = I - \frac{1}{m}J, \quad E_iE_j = \delta_{ij}E_i.$$

Then $(E_0 - E_1)^2 = I$. We claim

$$\begin{bmatrix} E_0 - E_1 & 0 \\ 0 & I \end{bmatrix} A \begin{bmatrix} E_0 - E_1 & 0 \\ 0 & I \end{bmatrix} = A'$$

where the rows and columns are indexed by $C \cup D$.

Indeed,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Since the subgraph induced on C is regular (equitable), of valency k, A_{11} commutes with J, hence also with $E_0 - E_1$. Thus

$$(E_0 - E_1)A_{11}(E_0 - E_1) = A_{11}.$$

How about $A_{21}(E_0 - E_1)$?

A_{21} consists of three kinds of row vectors, all-one $\mathbf{1}$, half-one (say u), zero.

$$\begin{aligned}
\mathbf{1}(E_0 - E_1) &= \mathbf{1}E_0 = \mathbf{1}, \\
u(E_0 - E_1) &= u(2E_0 - I) = 2(uE_0) - u \\
&= 2(\frac{1}{2}\mathbf{1}) - u = \mathbf{1} - u, \\
0(E_0 - E_1) &= 0.
\end{aligned}$$

This realizes Godsil-McKay switching.
It is straightforward to generalize to the case $C = \bigcup_i C_i$ consists of more than one C_i's.

Ferdinand Ihringer and I came up with a variation of Godsil-McKay switching, but after presenting it in a conference, I was notified by Wei Wang that his group independently discovered it (and published it very recently, gave a talk in the same conference two days later).

Suppose $V(\Gamma) = C_1 \cup C_2 \cup D$, $m = |C_1| = |C_2|$.
This time we use the matrix

$$Q = \begin{bmatrix} E_1 & E_0 & 0 \\ E_0 & E_1 & 0 \\ 0 & 0 & I \end{bmatrix} \in O_{|V(\Gamma)|}(\mathbb{Q}).$$

The fact that this is an orthogonal matrix comes from $E_i E_j = \delta_{ij} E_i$.
In order for QAQ to become an adjacency matrix of a graph, we need to assume some properties of the partition, as in Godsil-McKay switching.

Theorem (WQH switching)

Let Γ be a graph whose vertex set is partitioned as $C_1 \cup C_2 \cup D$. Assume that $|C_1| = |C_2|$ and that $C_1 \cup C_2$ is an equitable partition of the induced subgraph on $C_1 \cup C_2$, and that all $x \in D$ satisfy one of the following:

1. $|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|$, *or*
2. $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$.

Construct a graph $\overline{\Gamma}$ from Γ by modifying the edges between $C_1 \cup C_2$ and D as follows:

$$\overline{\Gamma}(x) \cap (C_1 \cup C_2) = \begin{cases} C_1 \text{ if } \Gamma(x) \cap (C_1 \cup C_2) = C_2, \\ C_2 \text{ if } \Gamma(x) \cap (C_1 \cup C_2) = C_1, \\ \Gamma(x) \cap (C_1 \cup C_2) \text{ otherwise,} \end{cases}$$

Consider a polar space of rank at least 3, for example, $U(6, q)$-space. All subspaces are considered as a set of projective points.
$V(\Gamma) =$ the set of isotropic points, and two points are adjacent if they are orthogonal. It has an isotropic plane P. Let $L_1, L_2 \subseteq P$ be distinct lines, and set

$$C_1 = L_1 \backslash L_2, \quad C_2 = L_2 \backslash L_1, \quad D = V(\Gamma) \backslash (C_1 \cup C_2)$$

This partition satisfies the hypotheses of WQH switching.

Indeed, $C_1 \cup C_2$ is a complete subgraph since it is contained in an isotropic plane P. Thus $C_1 \cup C_2$ is an equitable partition.
Let $x \in D$.
If $x \in P^\perp$, then $C_1 \cup C_2 \subset P \subset x^\perp$, so x is adjacent to all of $C_1 \cup C_2$.
Otherwise, $x^\perp \cap P = L$ is a line. If $L = L_1$, then x is adjacent to all of C_1, none of C_2. Similar for $L = L_2$.
If $L \neq L_1, L_2$, then $|L \cap L_1| = |L \cap L_2| = 1$, so x has the same number $(= 1)$ of neighbors in C_1 and in C_2.