# Krein parameters of fiber-commutative coherent configurations

#### Akihiro Munemasa

Graduate School of Information Sciences
Tohoku University

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#### Association schemes

X: a finite set.

$$X imes X = igcup_{i=0}^d R_i$$
 (disjoint),  $A_i = ext{adjacency matrix of } R_i,$   $\mathcal{A} = ext{linear span of } A_0, A_1, \ldots, A_d,$ 

Commutative association scheme:

$$A_0=I,$$
  $\mathcal{A}= rac{\mathsf{commutative}}{\mathsf{closed}} ext{ subalgebra of } M_n(\mathbb{C})$ 

$$A_i A_j = \sum_{k=0}^d p_{ij}^k A_k, \,\, E_i \circ E_j = rac{1}{|X|} \sum_{k=0}^d q_{ij}^k E_k.$$

Krein parameters: $q_{ij}^k$  which are nonnegative.

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Krein parameters: $q_{ij}^k$ ?

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Krein parameters? Theory is analogous for fiber-commutative

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# Summary of results

	commutative	fiber-commutative
	association	coherent
	scheme	configuration
2nd	primitive	basis of
basis	idempotents	matrix units
Krein	scalars	matrices
parameters	$q_{ij}^k$	$Q_{ij}^k$
	unique	essentially unique
Krein condition	$q_{ij}^k \geq 0$	$Q_{ij}^k \succeq 0$
absolute	$\sum_{\substack{q_{ij}^k  eq 0}} m_k \\ \leq m_i m_j$	$\sum_k m_k \operatorname{rank} Q_{ij}^k \ \leq m_i m_j$
bound	$\leq m_i m_j$	$\leq m_i m_j$

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- closed under ordinary mult., Hadamard mult., transposition,
- $\bullet$  contains I, J.

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Fiber commutative, i.e., all  $\mathcal{A}_{\alpha\alpha}$  are commutative  $\implies \dim \mathcal{I}_k \cap \mathcal{A}_{\alpha\beta} = 0$  or 1, by Hobart–Williford (2014).

### Basis of a fiber-comm. coherent algebra

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and each summand has dimension 0 or 1,  $\mathcal{I}_k$  has a basis  $\{e_{\alpha\beta}\}$  where  $(\alpha,\beta)$  runs through the set

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which is of the form  $\Lambda \times \Lambda$  for some index set  $\Lambda$ , so that

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Similarly, we can define  $Q_{ij}^k$  (matrix of Krein parameters).

#### Krein condition

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Krein condition (for coherent configurations) asserts

$$\forall F \in \mathcal{P}(\mathcal{I}_i), \ orall F' \in \mathcal{P}(\mathcal{I}_j), \ F \circ F' \succeq 0$$

or equivalently  $(F \circ F')E_k \in \mathcal{P}(\mathcal{I}_k)$  for all k, where  $E_k : \mathcal{A} \to \mathcal{I}_k$  is the orthogonal projection.

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or equivalently  $(F \circ F')E_k \in \mathcal{P}(\mathcal{I}_k)$  for all k, where  $E_k : \mathcal{A} \to \mathcal{I}_k$  is the orthogonal projection. For a fixed k, we have chosen a basis  $\{e_{\alpha\beta} \mid \alpha, \beta \in \Lambda\}$  so that

$$\mathcal{P}(\mathcal{I}_k) = \{ \sum_{lpha,eta} z_{lphaeta} e_{lphaeta} \mid (z_{lphaeta}) \in \mathcal{P}(M_{\Lambda}(\mathbb{C})) \}.$$

#### Theorem

For a fiber-commutative coherent algebra  $\mathcal{A}=\bigoplus_k \mathcal{I}_k$ , where  $\mathcal{I}_k=\mathcal{A}E_k\cong M_{\Lambda}(\mathbb{C})=\langle e_{\alpha\beta}^k\mid \alpha,\beta\in\Lambda
angle$ , Krein condition

$$(F \circ F')E_k \succeq 0 \quad (\forall F \in \mathcal{P}(\mathcal{I}_i), \ \forall F' \in \mathcal{P}(\mathcal{I}_j))$$

is equivalent to

$$Q_{ij}^k \succeq 0$$
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where  $Q_{ij}^k$  is the "matrix of Krein parameters" defined by

$$e^i_{lphaeta}\circ e^j_{lphaeta}=rac{1}{ extsf{scalar}}{\sum_k}(Q^k_{ij})_{lphaeta}e^k_{lphaeta}.$$

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Moreover,  $Q_{ij}^k$  is essentially unique.

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If time permits, I will prove the special case; otherwise, thank you for listening. This is the end.

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Note  $J \circ J \circ Q = Q$ . This explains Hobart's observation:

In our applications ..., we use  $\mathbf{Z} = \mathbf{Z}' = \phi_s(\mathbf{J})$ , where  $\mathbf{J}$  is the all 1s matrix. Other choices do not produce any new results for these particular examples.

Linear Algebra Appl. 226/228 (1995), p. 502.

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