Krein parameters of fiber-commutative coherent configurations

Akihiro Munemasa

Graduate School of Information Sciences
Tohoku University

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Association schemes

X: a finite set.

$$X \times X = \bigcup_{i=0}^{d} R_i \quad \text{(disjoint)},$$

$$A_i = \text{adjacency matrix of } R_i,$$

$$\mathcal{A} = \text{linear span of } A_0, A_1, \dots, A_d,$$

Commutative association scheme:

$$A_0 = I,$$

$$\mathcal{A} = \text{commutative subalgebra of } M_n(\mathbb{C})$$

closed under transposition

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k, \quad E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k.$$

Krein parameters: q_{ij}^k which are nonnegative.

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Krein parameters: q_{ij}^k ?

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Krein parameters? Theory is analogous for fiber-commutative

Summary of results

	commutative association scheme	fiber-commutative coherent configuration
2nd basis	primitive idempotents	basis of matrix units
Krein parameters	scalars q_{ij}^k unique	matrices Q_{ij}^k essentially unique
Krein condition	$q_{ij}^k \geq 0$	$Q_{ij}^k \succeq 0$
absolute bound	$\sum_{q_{ij}^k \neq 0} m_k \leq m_i m_j$	$\sum_k m_k \operatorname{rank} Q_{ij}^k \leq m_i m_j$

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- closed under ordinary mult., Hadamard mult., transposition,
- contains I, J.

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Fiber commutative, i.e., all $\mathcal{A}_{\alpha\alpha}$ are commutative
$\implies \dim \mathcal{I}_k \cap \mathcal{A}_{\alpha\beta} = 0$ or 1, by Hobart–Williford (2014).

Basis of a fiber-comm. coherent algebra

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which is of the form $\Lambda \times \Lambda$ for some index set Λ, so that

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Similarly, we can define Q^k_{ij} (matrix of Krein parameters).

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$$\forall F \in \mathcal{P}(\mathcal{I}_i),\ \forall F' \in \mathcal{P}(\mathcal{I}_j),\ F \circ F' \succeq 0$$

or equivalently $(F \circ F')E_k \in \mathcal{P}(\mathcal{I}_k)$ for all k, where $E_k : \mathcal{A} \to \mathcal{I}_k$ is the orthogonal projection.

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For a fixed k, we have chosen a basis $\{e_{\alpha\beta} \mid \alpha, \beta \in \Lambda\}$ so that

$$\mathcal{P}(\mathcal{I}_k) = \{\sum_{\alpha,\beta} z_{\alpha\beta} e_{\alpha\beta} \mid (z_{\alpha\beta}) \in \mathcal{P}(M_\Lambda(\mathbb{C}))\}.$$

Theorem

For a fiber-commutative coherent algebra $\mathcal{A} = \bigoplus_k \mathcal{I}_k$, *where* $\mathcal{I}_k = \mathcal{A}E_k \cong M_\Lambda(\mathbb{C}) = \langle e^k_{\alpha\beta} \mid \alpha, \beta \in \Lambda \rangle$, *Krein condition*

$$(F \circ F')E_k \succeq 0 \quad (\forall F \in \mathcal{P}(\mathcal{I}_i),\ \forall F' \in \mathcal{P}(\mathcal{I}_j))$$

is equivalent to

$$Q^k_{ij} \succeq 0,$$

where Q^k_{ij} *is the "matrix of Krein parameters" defined by*

$$e^i_{\alpha\beta} \circ e^j_{\alpha\beta} = \frac{1}{\text{scalar}} \sum_k (Q^k_{ij})_{\alpha\beta} e^k_{\alpha\beta}.$$

Moreover, Q^k_{ij} *is essentially unique.*

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If time permits, I will prove the special case;
otherwise, thank you for listening. This is the end.

Proof for the case $i = j = k$

$$\mathcal{P}(\mathcal{I}_k) = \{\sum z_{\alpha\beta} e_{\alpha\beta} \mid (z_{\alpha\beta}) \in \mathcal{P}(M_\Lambda(\mathbb{C}))\}.$$

Recall $Q = Q^k_{kk} = (q_{\alpha\beta})$ is defined by

$$(e_{\alpha\beta} \circ e_{\alpha\beta}) E_k = q_{\alpha\beta} e_{\alpha\beta}.$$

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&(F \circ F')E_k \succeq 0 \quad (\forall F, F' \in \mathcal{P}(\mathcal{I}_k)) \\
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\iff &\sum z_{\alpha\beta} z'_{\alpha\beta} (e_{\alpha\beta} \circ e_{\alpha\beta}) E_k \succeq 0 \\
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Note $J \circ J \circ Q = Q$. This explains Hobart's observation:

In our applications ..., we use $Z = Z' = \phi_s(J)$, where J is the all 1s matrix. Other choices do not produce any new results for these particular examples.

Linear Algebra Appl. 226/228 (1995), p. 502.