Hermitian adjacency matrices of digraphs and root lattices over the Gaussian integers

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joint work with A. Gavrilyuk
September 22, 2019

The International Conference on Algebraic Combinatorics
Henan Polytechnic University

The spectrum of a graph

It means the multiset of eigenvalues of its adjacency matrix.

$$\mathrm{Spec}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \mathrm{Spec}(A_2) = \{1, -1\},$$

$$\mathrm{Spec}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} = \mathrm{Spec}(A_3) = \{\sqrt{2}, 0, -\sqrt{2}\},$$

$$\mathrm{Spec}\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\end{bmatrix} = \mathrm{Spec}(\tilde{A}_3) = \{2, [0]^2, -2\},$$

$$\mathrm{Spec}(\tilde{A}_4) = \{2, [2\cos\frac{2\pi}{5}]^2, [2\cos\frac{4\pi}{5}]^2\}.$$

A_n = path with n vertices $\quad \tilde{A}_n = n+1$-cycle

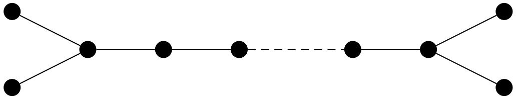
The spectral radius $\rho(\cdot)$ of a graph

Denote by $\rho(\cdot)$ the maximum of the absolute value of the spectrum of a graph.

$$\rho(A_2) = 1, \quad \rho(A_3) = \sqrt{2},$$
$$\rho(\tilde{A}_3) = 2, \quad \rho(\tilde{A}_4) = 2.$$

Smith (1970), Lemmens and Seidel (1974): Every graph with $\rho \leq 2$ is a subgraph of one of the following:

$$\tilde{A}_n = \text{ cycle}, \ \tilde{D}_n, \ \tilde{E}_6, \ \tilde{E}_7, \ \tilde{E}_8$$



$\tilde{D}_n$

Subgraphs are A_n, D_n, E_6, E_7, E_8, and these graphs have $\rho < 2$.

The Hermitian adjacency matrix $H = H(\Delta)$ of a digraph Δ, introduced by Liu–Li (2015), Guo–Mohar (2017):

$$H_{xy} = \begin{cases} 1 & \text{if } x \rightleftarrows y \\ i & \text{if } x \to y \\ -i & \text{if } x \leftarrow y \\ 0 & \text{otherwise} \end{cases}$$

Guo–Mohar (2017) classified digraphs with Hermitian spectral radius < 2. The result includes A_n, D_n, E_6, E_7, E_8.
Can we classify digraphs with Hermitian spectral radius $= 2$?
This will include $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.
In the undirected case, to go from "< 2" to "$= 2$", it suffice to add one vertex: from A_n (path) to $\tilde{A}_n$ (cycle).

Isomorphism and switching equivalence

Two undirected graphs G and G' with respective adjacency matrices A and A' are isomorphic if $\exists$ permutation matrix P such that

$$P^\top AP = A'.$$

Two digraphs Δ and Δ' with respective Hermitian adjacency matrices H and H' are switching equivalent if $\exists$ monomial matrix P with entries in $\{0, \pm 1, \pm i\}$ such that

$$P^* HP = H' \text{ or } P^* \overline{H} P = H'.$$

Guo–Mohar (2017) classified digraphs with Hermitian spectral radius < 2, up to switching equivalence.

Cyclotomic matrices classified by Greaves

Greaves (2012) classified maximal Hermitian matrices with

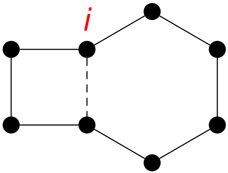
1. entries are in $\{0, 1, -1, i, -i\}$,
2. diagonals $= 0$,
3. spectral radius ≤ 2

up to equivalence: $H \sim H' \iff$

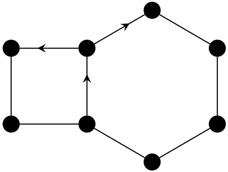
$$\exists \text{ monomial matrix } P,\ P^* H P = \pm H' \text{ or } P^* \overline{H} P = \pm H'.$$

Hermitian adjacency matrices of digraphs with $\rho \leq 2$ should all appear.
But they are mixed with matrices with -1 in its entries, due to weaker equivalence.

The dotted edge means a “minus” edge.

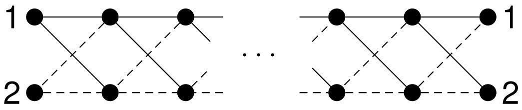


is equivalent to the digraph

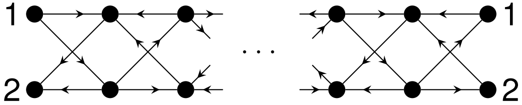


Toral tesselation: $\mathrm{Spec}(T_{2k}) = \{[2]^k, [-2]^k\}$

The signed graph T_{2k} (dotted edge means a "minus" edge)



is equivalent to the Hermitian adjacency matrix of the digraph Δ_{2k}:



Is there another digraph Δ such that $H(\Delta) \sim T_{2k}$ but Δ is not switching equivalent to Δ_{2k} above? (actually, no).
It seems difficult to classify all subdigraphs of Δ_{2k}.

Cameron–Goethals–Seidel–Shult (1976)

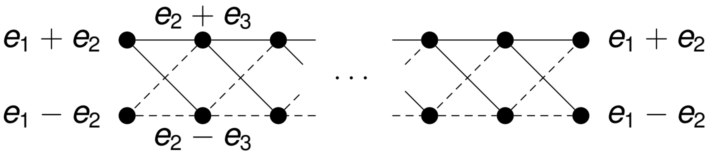
Every graph with $\lambda_{\min} \geq -2$ can be represented by a root system of type A_n, D_n or E_6, E_7, E_8.

$A + 2I$ is positive semidefinite, so it is the Gram matrix of a set of vectors of norm 2.

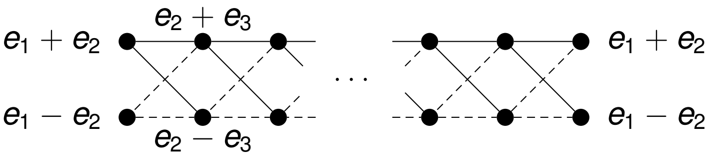
$$D_n = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}.$$

$T_{2k} + 2I$ is positive semidefinite. Indeed, represented by

$$\{e_p \pm e_{p+1} \mid 1 \leq p \leq k\},$$

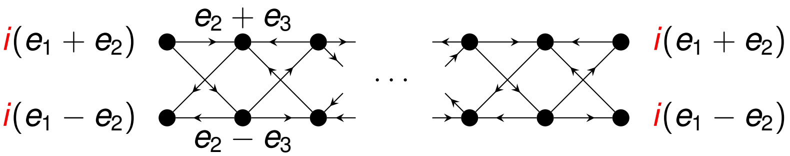


From T_{2k} to Δ_{2k} (k even)



The digraph Δ_{2k} is represented by

$$\{e_p \pm e_{p+1} \mid p \text{ even}\} \cup \{i(e_p \pm e_{p+1}) \mid p \text{ odd}\}$$



(The case k odd is slightly more complicated.)

Classification

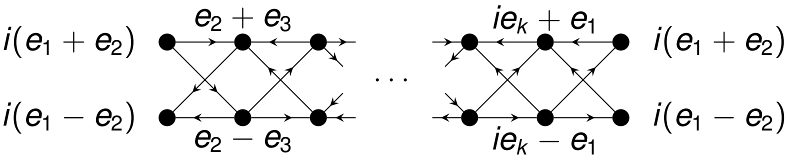
Theorem

Let Δ be a connected digraph with $\rho(\Delta) \leq 2$. Then Δ is switching equivalent to a subdigraph of: (all $\rho(\Delta) = 2$)

1. Δ_{2k}, $\Delta_{2k}^{(i)}$,
2. one of the three "exceptional" digraphs (8, 14, 16 vertices).

The digraph $\Delta_{2k}^{(i)}$ (k odd) is represented by

$$\{e_p \pm e_{p+1} \mid p \text{ even}\} \cup \{i(e_p \pm e_{p+1}) \mid p \text{ odd}\} \cup \{ie_k \pm e_1\}$$



(The case k even is slightly more complicated.)

Can we recover Guo–Mohar classification?

1. Our result relies on Greaves's classification: $\rho \leq 2$ & "maximal"
2. In principle, if we consider all subdigraphs, we should be able to recover...
3. McKee–Smyth (2007) classified signed graphs with $\rho < 2$.

A signed graph is a graph with edge weight $+1$ or -1. The adjacency matrix is then a $(0, \pm 1)$ matrix.

- Switching equivalence = conjugation by a $(0, \pm 1)$ monomial matrix

The associated signed graph $S(\Delta)$ of a digraph Δ:

$$H(\Delta) = A + iB \quad (A = A^\top,\ B = -B^\top) \implies A(S(\Delta)) = \begin{bmatrix} A & B \\ B^\top & A \end{bmatrix}$$

- $\operatorname{Spec} H(\Delta)^{\times 2} = \operatorname{Spec} A(S(\Delta))$, so $\rho(\Delta) = \rho(S(\Delta))$.

Gaussian root lattice

$$
\begin{array}{ccc}
\Delta & \xrightarrow{S} & S(\Delta) \\
\downarrow & & \downarrow \\
H(\Delta) = A + Bi & \longrightarrow & \begin{bmatrix} A & B \\ B^\top & A \end{bmatrix}
\end{array}
$$

$$
\begin{array}{ccc}
\text{Gaussian lattice} & \longleftarrow & \text{Euclidean lattice} \\
H(\Delta) + 2I & & \begin{bmatrix} A & B \\ B^\top & A \end{bmatrix} + 2I
\end{array}
$$

A lattice is

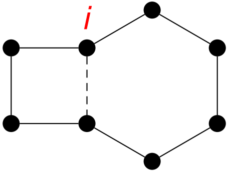
Euclidean:free $\mathbb{Z}$-module with positive definite sym. bil. form

Gaussian:free $\mathbb{Z}[i]$-module with positive definite Hermitian form

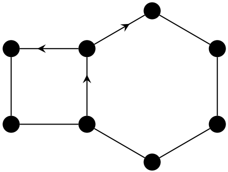
This Euclidean lattice is a (not necessarily irreducible) root

A digraph with $\rho < 2$

The signed graph (in McKee–Smyth)



is equivalent to the digraph



This digraph is missing in the Guo–Mohar classification.

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