Hoffman's Limit Theorem

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History ($\lambda_{\min}$ = smallest eigenvalue)

$$\lim_{t\to\infty} \lambda_{\min} \begin{bmatrix} A & C \otimes \mathbf{1}_t \\ C^\top \otimes \mathbf{1}_t^\top & I \otimes (J_t - I_t) \end{bmatrix} = \lambda_{\min}(A - CC^\top).$$

- Hoffman (SIAM, 1969) stated a theorem (Hoffman's limit theorem), "is shown in [4]" where [4]=Hoffman & Ostrowski, "to appear" was never published.
- Hoffman (LAA, 1977), citing above, proved a theorem about graphs with $\lambda_{\min} \in (-2, -1)$ and $\lambda_{\min} \in (-1 - \sqrt{2}, -2)$.
- Jang–Koolen–M.–Taniguchi (AMC, 2014) gave a graph theoretic proof.
- Hoffman (Geom. Ded. 1977), proved signed graph version of the limit theorem.

Today, we give a Hermitian matrix version of the limit theorem and an application to signed graphs with $\lambda_{\min} \in (-2, -1)$.

What is the spectrum of a graph

The spectrum of a graph means the multiset of eigenvalues of its adjacency matrix.

$$\mathrm{Spec}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \{1, -1\},$$

$$\mathrm{Spec}(K_n) = \mathrm{Spec}(J_n - I_n) = \{[n-1]^1, [-1]^{n-1}\},$$

$$\mathrm{Spec}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \{\sqrt{2}, 0, -\sqrt{2}\},$$

$$\mathrm{Spec}\begin{bmatrix} 0 & 0 & \mathbf{1}_t \\ 0 & 0 & \mathbf{1}_t \\ \mathbf{1}_t^\top & \mathbf{1}_t^\top & J_t - I_t \end{bmatrix} = ?$$

$\rightarrow$ on blackboard

The smallest eigenvalue of a graph

Denote by $\lambda_{\min}(\cdot)$ the smallest eigenvalue of a matrix or a graph.

$$\lambda_{\min}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = -1,$$

$$\lambda_{\min}(K_n) = \lambda_{\min}(J_n - I_n) = -1,$$

$$\lambda_{\min}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} = -\sqrt{2},$$

$$\lambda_{\min}\begin{bmatrix}0 & 0 & \mathbf{1}_t\\ 0 & 0 & \mathbf{1}_t\\ \mathbf{1}_t^\top & \mathbf{1}_t^\top & J_t - I_t\end{bmatrix} = ?$$

Γ is connected and $\lambda_{\min}(\Gamma) = -1 \implies \Gamma \cong K_n$.

$$\operatorname{Spec}\begin{bmatrix} 0 & 0 & \mathbf{1}_t \\ 0 & 0 & \mathbf{1}_t \\ \mathbf{1}_t^\top & \mathbf{1}_t^\top & J_t - I_t \end{bmatrix} = \operatorname{Spec}\begin{bmatrix} 0 & 0 & t \\ 0 & 0 & t \\ 1 & 1 & t-1 \end{bmatrix} \cup \operatorname{Spec}(-I_{t-1}).$$

$$\begin{aligned} \lambda_{\min}\begin{bmatrix} 0 & 0 & \mathbf{1}_t \\ 0 & 0 & \mathbf{1}_t \\ \mathbf{1}_t^\top & \mathbf{1}_t^\top & J_t - I_t \end{bmatrix} &= \lambda_{\min}\begin{bmatrix} 0 & 0 & t \\ 0 & 0 & t \\ 1 & 1 & t-1 \end{bmatrix} \\ &= \min\{z \mid z(z^2 - (t-1)z - 2t) = 0\} \\ &= \frac{t - 1 - \sqrt{t^2 + 6t + 1}}{2} \to -2 \quad (t \to \infty). \end{aligned}$$

Shortcut (?)

$$\min\{z \mid (z+2) - \frac{1}{t}(z^2 + z) = 0\} \to \min\{z \mid z + 2 = 0\} = -2.$$

Hurwitz's theorem

Rahman & Schmeisser, "Analytic Theory of Polynomials," Theorem 1.3.8

Theorem

Let $(f_t)_{t=1}^{\infty}$ be a sequence of analytic functions defined in a region $\Omega \subseteq \mathbb{C}$. Suppose

$$f_t \to f \neq 0 \quad (t \to \infty)$$

uniformly on every compact subset of Ω. Then for $\zeta \in \Omega$, the following are equivalent:

1. ζ is a zero of f of multiplicity m,
2. $\zeta \in \exists U \subseteq \Omega$ (neighbourhood), $\forall \varepsilon > 0$, $\exists n_0 < \forall t$, f_t has exactly m zeros in the ε-neighbourhood of ζ.

Theorem (Hoffman's limit theorem)

Let

$$\begin{bmatrix} A & C \\ C^\top & 0 \end{bmatrix}$$

be the adjacency matrix of a graph. Then

$$\lim_{t\to\infty} \lambda_{\min} \begin{bmatrix} A & C \otimes \mathbf{1}_t \\ C^\top \otimes \mathbf{1}_t^\top & I \otimes (J_t - I_t) \end{bmatrix} = \lambda_{\min}(A - CC^\top).$$

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$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, A - CC^\top = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which has $\lambda_{\min} = -2$. Note

$$\lambda_{\min} \begin{bmatrix} 0 & 0 & \mathbf{1}_t \\ 0 & 0 & \mathbf{1}_t \\ \mathbf{1}_t^\top & \mathbf{1}_t^\top & J_t - I_t \end{bmatrix} > -2.$$

Cameron–Goethals–Seidel–Shult (1976)

Every graph with $\lambda_{\min} \geq -2$ can be represented by a root system of type A_n, D_n or E_6, E_7, E_8.

$$A = \begin{bmatrix} 0 & 0 & \mathbf{1}_t \\ 0 & 0 & \mathbf{1}_t \\ \mathbf{1}_t^\top & \mathbf{1}_t^\top & J_t - I_t \end{bmatrix}, \quad \lambda_{\min}(A) > -2.$$

$$D_n = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}.$$

$$M = \begin{bmatrix} -1 & 1 & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & 1 & & 1 & \\ & \vdots & & & \ddots \end{bmatrix}, \; MM^\top = A + 2I$$

Row vectors of M are in the root system D_n.

Proof of Hoffman's limit theorem

$A : n \times n$, $C : n \times m$.

$$\lim_{t\to\infty} \lambda_{\min} \begin{bmatrix} A & C \otimes \mathbf{1}_t \\ C^\top \otimes \mathbf{1}_t^\top & I \otimes (J_t - I_t) \end{bmatrix} = \lim_{t\to\infty} \lambda_{\min} \begin{bmatrix} A & tC \\ C^\top & (t-1)I \end{bmatrix}$$

Since

$$\begin{aligned} &\left| zI - \begin{bmatrix} A & tC \\ C^\top & (t-1)I \end{bmatrix} \right| \\ &= t^n (z+1-t)^{m-n} \left| A - CC^\top - zI + \frac{z+1}{t}(zI - A) \right|, \end{aligned}$$

the spectrum containing $\lambda_{\min} \to \mathrm{Spec}(A - CC^\top)$.
$\lambda_{\min} \to \lambda_{\min}(A - CC^\top)$, proving the theorem.

The same proof shows the Hermitian matrix version:

Theorem

Let

$$\begin{bmatrix} A & C \\ C^* & 0 \end{bmatrix}$$

be a Hermitian matrix, and let D be a positive definite Hermitian matrix. Then

$$\lim_{t\to\infty} \lambda_{\min} \begin{bmatrix} A & C \otimes \mathbf{1}_t \\ C^* \otimes \mathbf{1}_t^\top & D \otimes (J_t - I_t) \end{bmatrix} = \lambda_{\min}(A - CD^{-1}C^*).$$

A signed graph is a graph with edge weight $+1$ or -1. The adjacency matrix is then a $(0, \pm 1)$ matrix.

- Switching equivalence = conjugation by a $(0, \pm 1)$ monomial matrix
- $\delta(G) :=$ minimum degree of G.

Theorem

There exists a function $f : (-2, -1) \to \mathbb{R}$ such that, for each $\lambda \in (-2, -1)$, if G is a connected signed graph with $\lambda_{\min}(G) \geq \lambda$, $\delta(G) \geq f(\lambda)$, then G is switching equivalent to a complete graph.

The proof is a simplification of Hoffman's original by incorporating Cameron–Goethals–Seidel–Shult (1976), Greaves–Koolen–M.–Sano–Taniguchi (2015).

Proof (part 1)

Fix $\lambda \in (-2, -1)$. To prove this theorem, it suffices to show that,

$$\begin{array}{l} \lambda_{\min}(G) \geq \lambda \\ \delta(G) \text{ sufficiently large} \end{array} \implies G \text{ is sw. eq. } K_n.$$

By Cameron–Goethals–Seidel–Shult (1976), we may assume G is represented by A_m or D_m (ignoring E_6, E_7, E_8).

But $A_m \subseteq D_{m+1}$, so

Proof (part 2)

$$\left.\begin{array}{l}\lambda_{\min}(G) \geq \lambda \\ \delta(G) \text{ sufficiently large} \\ G \text{ is represented by } D_m\end{array}\right. \implies G \text{ is sw. eq. } K_n.$$

Greaves–Koolen–M.–Sano–Taniguchi (2015) classified such signed graphs. In particular,

Theorem

Let G be a connected signed graph represented by D_m and $\lambda_{\min}(G) > -2$. Then there exists a tree T such that the line graph $L(T)$ of T is switching equivalent to G with possibly one vertex removed.

We illustrate the proof for the case $G = L(T)$.

$$\left.\begin{array}{l}\lambda_{\min}(L(T)) \geq \lambda \\ \delta(L(T)) \text{ sufficiently large}\end{array}\right. \implies L(T) \cong K_n.$$

Recall the Hermitian adjacency matrix $H = H(\Delta)$ of a digraph Δ:

$$H_{xy} = \begin{cases} 1 & \text{if } x \rightleftarrows y \\ i & \text{if } x \to y \\ -i & \text{if } x \leftarrow y \\ 0 & \text{otherwise} \end{cases}$$

Introduced by Liu–Li (2015), Guo–Mohar (2017).

Theorem

There exists a function $f : (-2, -1) \to \mathbb{R}$ such that, for each $\lambda \in (-2, -1)$, if Δ is a connected digraph with $\lambda_{\min}(H(\Delta)) \geq \lambda$, $\delta(\overline{\Delta}) \geq f(\lambda)$, then Δ is switching equivalent to a complete graph.

- Switching equivalence = conjugation by a $(0, \pm 1, \pm i)$ monomial matrix, and possibly taking the transpose
- $\delta(\overline{\Delta}) :=$ minimum degree of the underlying undirected graph of Δ.

Theorem

There exists a function $f : (-2, -1) \to \mathbb{R}$ such that, for each $\lambda \in (-2, -1)$,

1. for connected signed graph G, $\lambda_{\min}(G) \geq \lambda$, $\delta(G) \geq f(\lambda) \implies G$ sw. eq. K_n.
2. for connected digraph Δ, $\lambda_{\min}(H(\Delta)) \geq \lambda$, $\delta(\overline{\Delta}) \geq f(\lambda) \implies \Delta$ sw. eq. K_n.

The digraph version is immediate from signed graph version by considering the associated signed graph:

$$H(\Delta) = A + iB \quad (A = A^\top,\ B = -B^\top) \implies A(G) = \begin{bmatrix} A & B \\ B^\top & A \end{bmatrix}$$

- Spec $H(\Delta)^{\times 2} =$ Spec G, so $\lambda_{\min} H(\Delta) = \lambda_{\min} G$.
- $\delta(\overline{\Delta}) = \delta(G)$.

Further results yet to be generalized: Hoffman (1977): $(-1-\sqrt{2}, -2)$, Woo & Neumaier (1995).

The idea of associated signed graph comes from

- Masaaki Kitazume and A. M., Even unimodular Gaussian lattices of rank 12, J. Number Theory (2002).

Gaussian lattices of rank 12 $\leftrightarrow$ Euclidean lattices of rank 24

A lattice in $\mathbb{R}^n$ is a subgroup $L \subset \mathbb{R}^n$,

$$L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n.$$

for some basis $\{e_1, \cdots, e_n\}$ of $\mathbb{R}^n$. The dual L is

$$L^\sharp = \{y \in \mathbb{R}^n \mid (y, x) \in \mathbb{Z},\ \forall x \in L\},$$

A lattice L is called

$$\begin{aligned} \text{integral if } (x, y) &\in \mathbb{Z} \text{ for all } x, y \in L, \\ \text{even if } (x, x) &\in 2\mathbb{Z} \text{ for all } x \in L, \\ \text{unimodular if } L^\sharp &= L. \end{aligned}$$

A digraph with n vertices $\to$ its associated signed graph has $2n$ vertices:

$$H(\Delta) = A + iB \quad (A = A^\top,\ B = -B^\top) \implies A(G) = \begin{bmatrix} A & B \\ B^\top & A \end{bmatrix}$$

- Given a positive definite symmetric matrix S with integer entries and diagonal 2, find a Hermitian matrix $H = A + iB$ with entries in $\{\pm 1, \pm i, 0\}$ such that

$$S \cong \begin{bmatrix} A & B \\ B^\top & A \end{bmatrix}$$

- Given a signed adjacency matrix S of order $2n$, find a Hermitian matrix $H = A + iB$ of order n such that S is switching equivalent to

$$\begin{bmatrix} A & B \\ B^\top & A \end{bmatrix}$$