Equiangular lines in Euclidean spaces

Akihiro Munemasa

Graduate School of Information Sciences
Tohoku University

joint work with G. Greaves, J. Koolen, and F. Szöllősi
September 28, 2019

Shanghai University

By a set of equiangular lines with angle $\arccos\alpha$ in $\mathbb{R}^d$, we mean

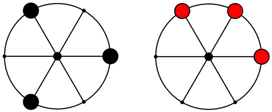
$$\{\mathbb{R}\boldsymbol{x}_1, \dots, \mathbb{R}\boldsymbol{x}_n\},$$

where $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d$ are unit vectors such that

$$|(\boldsymbol{x}_i, \boldsymbol{x}_j)| = \alpha \quad (1 \le i < j \le n),$$

and

$$0 \le \alpha < 1.$$



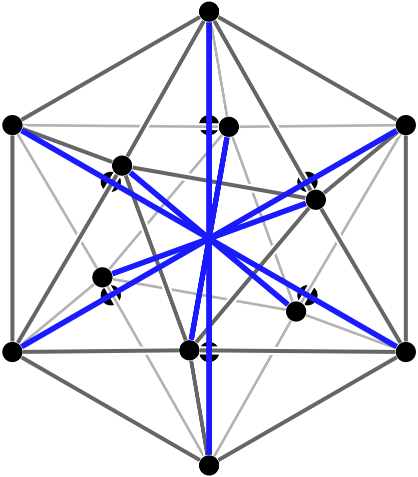
Example: $d = 2$, $\alpha = 1/2$,

$\boldsymbol{x}_k = (\cos\frac{2\pi k}{3}, \sin\frac{2\pi k}{3}) \quad (k = 1, 2, 3)$

$\boldsymbol{y}_k = (\cos\frac{\pi k}{3}, \sin\frac{\pi k}{3}) \quad (k = 0, 1, 2)$

12 vertices of the Icosahedron = 6 lines

Example: $d = 3$, $\alpha = 1/\sqrt{5}$, six diagonals of the icosahedron



$\arccos(1/\sqrt{5}) \sim 63^\circ$.

(illustration by Gary Greaves)

Set of points in $S^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \|\boldsymbol{x}\| = 1\}$

Equiangular lines:

$$(\boldsymbol{x}_i, \boldsymbol{x}_j) = \pm\alpha \quad (1 \le i < j \le n).$$

Maximize the number of lines n:

$$N_\alpha(d) = \max\{|X| \mid X \subseteq S^{d-1} \mid (\boldsymbol{x}, \boldsymbol{y}) = \pm\alpha \ (\forall \boldsymbol{x}, \boldsymbol{y} \in X,\ \boldsymbol{x} \neq \boldsymbol{y})\},$$
$$N(d) = \max\{N_\alpha(d) \mid 0 \le \alpha < 1\}.$$

A similar problem is the sphere packing (kissing number) problem:

$$\tau(d) = \max\{|X| \mid X \subseteq S^{d-1} \mid (\boldsymbol{x}, \boldsymbol{y}) \le \frac{1}{2} \ (\forall \boldsymbol{x}, \boldsymbol{y} \in X,\ \boldsymbol{x} \neq \boldsymbol{y})\}.$$

$N(2) = 3$, $\tau(2) = 6$ (hexagon)
$N(3) = 6$: Haantjes (1948).
$\tau(3) = 12$ (icosahedron): Schütte and van der Waerden (1953).

The value α

$$N(2) = N_{1/2}(2), \quad N(3) = N_{1/\sqrt{5}}(3).$$

For $d \geq 4$, for which $\alpha \in [0,1)$, $N(d) = N_\alpha(d)$ holds?

Theorem (Lemmens–Seidel, P. M. Neumann, 1973)

Suppose $\exists n$ equiangular lines with angle $\arccos\alpha$ in $\mathbb{R}^d$.

$$n > 2d \implies \frac{1}{\alpha} \text{ is an odd integer } \geq 3.$$

Is the hypothesis $n > 2d$ restrictive? No.

d	2	3	4	5	6	7–13	14	...
$N(d)$	3	6	6	10	16	28	?	...
$1/\alpha$	2	$\sqrt{5}$	$\sqrt{5}$ or 3	3	3	3	3 or 5	

$$N(d) = \Theta(d^2) \quad (d \to \infty).$$

$\alpha = 1/3$: Root systems

Suppose $\exists n$ equiangular lines with angle $\arccos(1/3)$ in $\mathbb{R}^d$. The Gram matrix

$$G = ((\boldsymbol{x}_i, \boldsymbol{x}_j))$$

has diagonal $= 1$, off diagonal $= \pm\frac{1}{3}$.
Let J denote the all-one matrix.

$$S = 3(G - I) \quad \text{(Seidel matrix): off diagonal} = \pm 1$$

$$A = \frac{1}{2}(J - I + S) \quad \text{(adjacency matrix): off diagonal} = 0, 1$$

$$C = A + 2I = \frac{1}{2}J + \frac{3}{2}G \geq 0.$$

C is the Gram matrix of a subset of a root system of type A, D, E.

Van Lint–Seidel (1966):

$$N_\alpha(d) \le 1 + \frac{d-1}{1-d\alpha^2} \quad \text{if } 1 - d\alpha^2 > 0.$$

d	3	4	5	6	7
$N_{1/3}(d)$	4	6	10	16	28

$\arccos \frac{1}{3} \sim 70^\circ$

Lemmens–Seidel (1973):

d	3	4	5	6	7–13	14	15	16–
$N_{1/3}(d)$	4	6	10	16	28	28	28	$2(d-1)$
$N(d)$	6	6	10	16	28	?	$36 = N_{1/5}$	?

$$N_{1/5}(14) \le \frac{336}{11} = 30.5\ldots, \qquad N_{1/7}(14) \le 19, \ldots.$$

Tremain (2008): $28 \le N_{1/5}(14)$.
Thus

$$28 \le N_{1/5}(14) = N(14) \le 30.$$

$N(14)$

$$N(14) = N_{1/5}(14) = 28 \text{ or } 29 \text{ or } 30.$$

Theorem (Greaves–Koolen–M.–Szöllősi, 2016)

$N_{1/5}(14) < 30.$

So

$$N(14) = N_{1/5}(14) = 28 \text{ or } 29.$$

Our method is not powerful enough to rule out 29.

$N_{1/5}(14)$

Suppose $\exists n$ equiangular lines with angle $\arccos(1/5)$ in $\mathbb{R}^d$. The Gram matrix

$$G = ((\boldsymbol{x}_i, \boldsymbol{x}_j))$$

has diagonal $= 1$, off diagonal $= \pm\frac{1}{5}$, rank $G = d$.

$$S = 5(G - I) \quad \text{(Seidel matrix): off diagonal} = \pm 1$$

$$A = \frac{1}{2}(J - I + S) \quad \text{(adjacency matrix): off diagonal} = 0, 1$$

$$C = A + 3I = \frac{1}{2}J + \frac{5}{2}G \geq 0.$$

C is the Gram matrix of a set of vectors of norm 3, with inner products $0, 1$, in $\mathbb{R}^{d+1}$.

Future work

Root systems:

- The set of vectors of a lattice generated by norm 2 vectors, with inner products $0, \pm 1$.
- Classified by Cartan, Killing, Witt.
- Denoted by A_d ($d \geq 2$), D_d ($d \geq 4$), E_d ($d = 6, 7, 8$).

Sets of vectors of norm 3 with inner products $0, \pm 1$ (no name)

- Such a set generates an integral lattice.
- Classification(?)

Theorem (Conway–Sloane, 1989)

Every integral lattice of rank r can be embedded in a unimodular lattice of rank at most $r + 3$.

- Classification of unimodular lattices is available for $d \leq 25$. In particular, for rank $(14 + 1) + 3 = 18$.