Extremal Finite Sets in Spheres and Projective Spaces

Akihiro Munemasa

Graduate School of Information Sciences
Tohoku University

January 6, 2020
ISI Kolkata

About me

- Name: Akihiro Munemasa
- Affiliation: Tohoku University, Sendai, Japan
- Specialization: Algebra & Combinatorics
- Ph.D (1989) from The Ohio State University, under E. Bannai.

My dissertation committee includes: Dijen K. Ray-Chaudhuri.

Raj Chandra Bose
|
Dijen K. Ray-Chaudhuri E. Bannai
| ⋱ |
K. T. Arasu A. M.

In the unit sphere $S^{d-1} \subseteq \mathbb{R}^d$

Extremal finite sets in S^{d-1} can mean:

(a) Large finite set with few distances or large enough mutual distances

(b) Small finite set which approximates the sphere well

The theory of spherical design (in an appropriate setting):

$$\begin{aligned} &\text{maximizing the size of a set in (a)} \\ &= \text{minimizing the size of a set in (b)} \end{aligned}$$

(a) is similar to coding theory: Large rate (number of codewords) with large minimum distance.

Equiangular lines

By a set of equiangular lines with angle $\arccos\alpha$ in $\mathbb{R}^d$, we mean

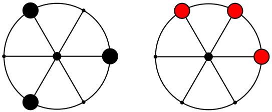
$$\{\mathbb{R}\boldsymbol{x}_1, \dots, \mathbb{R}\boldsymbol{x}_n\},$$

where $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d$ are unit vectors such that

$$|(\boldsymbol{x}_i, \boldsymbol{x}_j)| = \alpha \quad (1 \le i < j \le n),$$

and

$$0 \le \alpha < 1.$$



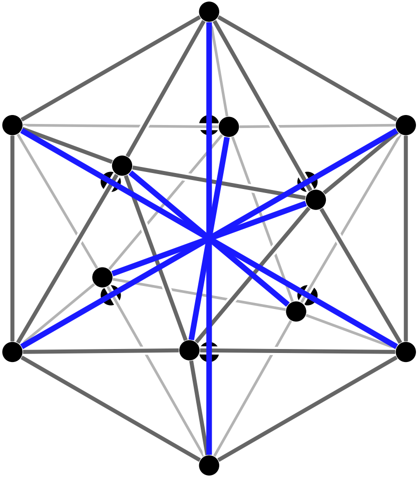
Example: $d = 2$, $\alpha = 1/2$,

$$\boldsymbol{x}_k = (\cos\frac{2\pi k}{3}, \sin\frac{2\pi k}{3}) \quad (k = 1, 2, 3)$$

$$\boldsymbol{y}_k = (\cos\frac{\pi k}{3}, \sin\frac{\pi k}{3}) \quad (k = 0, 1, 2)$$

12 vertices of the Icosahedron = 6 lines

Example: $d = 3$, $\alpha = 1/\sqrt{5}$, six diagonals of the icosahedron



$\arccos(1/\sqrt{5}) \sim 63^\circ$.

(illustration by Gary Greaves)

Set of points in $S^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \|\boldsymbol{x}\| = 1\}$

Equiangular lines:

$$(\boldsymbol{x}_i, \boldsymbol{x}_j) = \pm\alpha \quad (1 \le i < j \le n).$$

Maximize the number of lines n:

$$N_\alpha(d) = \max\{|X| \mid X \subseteq S^{d-1} \mid (\boldsymbol{x}, \boldsymbol{y}) = \pm\alpha\ (\forall \boldsymbol{x}, \boldsymbol{y} \in X,\ \boldsymbol{x} \ne \boldsymbol{y})\},$$
$$N(d) = \max\{N_\alpha(d) \mid 0 \le \alpha < 1\}.$$

A similar problem is the sphere packing (kissing number) problem:

$$\tau(d) = \max\{|X| \mid X \subseteq S^{d-1} \mid (\boldsymbol{x}, \boldsymbol{y}) \le \frac{1}{2}\ (\forall \boldsymbol{x}, \boldsymbol{y} \in X,\ \boldsymbol{x} \ne \boldsymbol{y})\}.$$

$N(2) = 3$, $\tau(2) = 6$ (hexagon)
$N(3) = 6$: Haantjes (1948).
$\tau(3) = 12$ (icosahedron): Schütte and van der Waerden (1953).

The value α in $N_\alpha(d)$

$$N(2) = N_{1/2}(2), \quad N(3) = N_{1/\sqrt{5}}(3).$$

For $d \geq 4$, for which $\alpha \in [0, 1)$, $N(d) = N_\alpha(d)$ holds?

Theorem (Lemmens–Seidel, P. M. Neumann, 1973)

Suppose $\exists n$ equiangular lines with angle $\arccos \alpha$ in $\mathbb{R}^d$.

$$n > 2d \implies \frac{1}{\alpha} \quad \text{is an odd integer } \geq 3.$$

Is the hypothesis $n > 2d$ restrictive? No.

d	2	3	4	5	6	7–13	14	...
$N(d)$	3	6	6	10	16	28	?	...
$1/\alpha$	2	$\sqrt{5}$	$\sqrt{5}$ or 3	3	3	3	3 or 5	

$$N(d) = \Theta(d^2) \quad (d \to \infty).$$

$1/\alpha = 3$: Root systems

Suppose $\exists n$ equiangular lines with angle arccos$(1/3)$ in $\mathbb{R}^d$. The Gram matrix

$$G = ((\boldsymbol{x}_i, \boldsymbol{x}_j))$$

has diagonal $= 1$, off diagonal $= \pm\frac{1}{3}$.
Let J denote the all-one matrix.

$$S = 3(G - I) \quad \text{(Seidel matrix): off diagonal} = \pm 1$$

$$A = \frac{1}{2}(J - I + S) \quad \text{(adjacency matrix): off diagonal} = 0, 1$$

$$C = A + 2I = \frac{1}{2}J + \frac{3}{2}G \geq 0.$$

C is the Gram matrix of a subset of a root system of type A, D, E.

Van Lint–Seidel (1966):

$$N_\alpha(d) \le 1 + \frac{d-1}{1-d\alpha^2} \quad \text{if } 1 - d\alpha^2 > 0.$$

d	3	4	5	6	7
$N_{1/3}(d)$	4	6	10	16	28

$\arccos \frac{1}{3} \sim 70^\circ$

Lemmens–Seidel (1973):

d	3	4	5	6	7–13	14	15	16–
$N_{1/3}(d)$	4	6	10	16	28	28	28	$2(d-1)$
$N(d)$	6	6	10	16	28	?	$36 = N_{1/5}$	?

$$N_{1/5}(14) \le \frac{336}{11} = 30.5\ldots, \quad N_{1/7}(14) \le 19, \ldots.$$

Tremain (2008): $28 \le N_{1/5}(14)$.

Thus

$$28 \le N_{1/5}(14) = N(14) \le 30.$$

$N(14)$

$$N(14) = N_{1/5}(14) = 28 \text{ or } 29 \text{ or } 30.$$

Theorem (Greaves–Koolen–M.–Szöllősi, 2016)

$N_{1/5}(14) < 30.$

So

$$N(14) = N_{1/5}(14) = 28 \text{ or } 29.$$

Our method is not powerful enough to rule out 29.

The upper bound of $N_\alpha(d)$ for $\alpha \leq 1/\sqrt{d+2}$

$$N_\alpha(d) \leq 1 + \frac{d-1}{1-d\alpha^2} \qquad (1)$$

For a set $X = \{\mathbb{R}\boldsymbol{x}_1, \dots, \mathbb{R}\boldsymbol{x}_n\}$ of equiangular lines with mutual angle $\arccos\alpha$, the following are equivalent:

1. X achieves the above upper bound
2. X is a tight frame
3. $\{\pm\boldsymbol{x}_1, \dots, \pm\boldsymbol{x}_n\}$ is a spherical 2-design

Moreover, for $\alpha = 1/\sqrt{d+2}$, the bound is the largest:

$$N_\alpha(d) \leq N_{1/\sqrt{d+2}}(d) = \frac{d(d+1)}{2}. \qquad (2)$$

Equality in (2) is achieved if and only if X is a spherical 4-design.

Tight frames and spherical designs

A set of unit vectors $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\} \subseteq \mathbb{R}^d$ is a tight frame if $\exists c \neq 0$,

$$\boldsymbol{x} = c \sum_{i=1}^{n} (\boldsymbol{x}, \boldsymbol{x}_i) \boldsymbol{x}_i \quad (\forall \boldsymbol{x} \in \mathbb{R}^d).$$

X is called a spherical t-design if

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \int_{S^{d-1}} f(x) d\sigma(x)$$

for all polynomial functions $f(x)$ of degree at most t.

Complex tight frames

Let H be a Hilbert space. A set of unit vectors $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\} \subseteq H$ is called a tight frame if $\exists c \neq 0$,

$$\boldsymbol{x} = c \sum_{i=1}^{n} (\boldsymbol{x}, \boldsymbol{x}_i) \boldsymbol{x}_i \quad (\forall \boldsymbol{x} \in H).$$

If H is over $\mathbb{C}$, we say that H is equiangular if

$$|(\boldsymbol{x}_i, \boldsymbol{x}_j)| \text{ is constant independent of } i \neq j$$

Zauner's conjecture (SIC-POVM): $\exists$ an equiangular tight frame of size d^2 in $\mathbb{C}^d$, with

$$|(\boldsymbol{x}_i, \boldsymbol{x}_j)| = \frac{1}{d+1} \quad (i \neq j).$$

Gerzon bound on $N(d)$

$$N(d) = \text{the largest size of a set of equiangular lines in } d\text{-space}$$
$$\leq \begin{cases} \frac{1}{2}d(d+1) & \text{over } \mathbb{R}, \\ d^2 & \text{over } \mathbb{C}. \end{cases}$$

The upper bound is believed to be achieved for $\mathbb{C}$ (Zauner's conjecture on SIC-POVM).
The upper bound for $\mathbb{R}$ is achieved for $d = 2, 3, 7, 23$ and possibly $d = (2m+1)^2 - 2$ ($m \in \mathbb{N}$).
When the bound is achieved with $d = (2m+1)^2 - 2$,

$$N(d) = N_\alpha(d) \text{ with } \alpha = \frac{1}{2m+1}.$$

and the set gives a spherical 4-design.

Gerzon bound on $N(d)$ over R

$$N(d) \leq \frac{d(d+1)}{2}.$$

If equality holds and $d > 3$, then $d = (2m+1)^2 - 2$ for some m.
$m = 1 \implies d = 7 \implies$ unique (a hyperplane in E_8 root system).
$m = 2 \implies d = 23 \implies$ unique.

- Makhnev (2002) ruled out $m = 3$
- Bannai–M.–Venkov (2004) ruled out $m = 3, 4$ and infinitely many others
- Nebe–Venkov (2011) ruled out $m = 6$ and infinitely many others

Still open: $m = 5$, i.e., $d = 119$.
Thank you very much for your attention.