# The regular two-graph on 276 vertices revisited

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joint work with Jack Koolen

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- Goethals–Seidel (1975) "The regular two-graph on 276 vertices"
- Godsil–Royle "Algebraic Graph Theory"
  - Chapter 11 "Two-Graphs"
  - Section 11.8 "The Two-Graph on 276 vertices"
- Two-graph $=$ Switching class of graphs
- McLaughlin (1969): Sporadic finite simple group $McL$ acting on a graph with 275 vertices (McLaughlin graph).
- $M_{22} \leq McL \leq Co._3$; Mathieu (1873), Conway (1968–1969).

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Switching of $\Gamma = (V, E)$ with respect to $U \subseteq V$ is $\Gamma^U = (V, E^U)$, where

$$\begin{aligned} E^U = &\{\{x,y\} \in E : x, y \in U\} \\ &\cup \{\{x,y\} \in E : x, y \in V \setminus U\} \\ &\cup \{\{x,y\} \notin E : x \in U,\ y \in V \setminus U\}. \end{aligned}$$

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$$\{\Gamma^U : U \subseteq V\}.$$

It consists of $2^{|V|-1}$ graphs, since

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Let $\Gamma = L(K_8)$: line graph of $K_8$, is strongly regular with parameters

$$SRG(28, 12, 6, 4).$$

$$V = V(L(K_8)) \\ = \{e_i + e_j : 1 \leq i < j \leq 8\} \subseteq \mathbb{R}^8.$$

For $u, v \in V$,

$$u \sim v \iff (u, v) = 1.$$

The switching class of $\Gamma$ contains $K_1 \cup$Sch, where Sch is the Schläfli graph

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$$r = \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1),$$
$$H = \{x \in \mathbb{R}^8 : (r, x) = 1\}.$$

Then

$$\begin{aligned} (r, r) &= 2, \\ V &= \{e_i + e_j : 1 \le i < j \le 8\} \subseteq H. \end{aligned}$$

In fact, $V \cup \{r\}$ is a part of the $E_8$ root system,

$$H \cap E_8 = V \cup \{r - u : u \in V\}.$$

Write

$$\overline{u} = u - \frac{1}{2}r \quad (u \in V).$$

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gives a set of $28$ equiangular lines in $H \cong \mathbb{R}^7$.

$$(u,v) = \begin{cases} 1 \\ 0 \end{cases} \iff (\overline{u},\overline{v}) = \begin{cases} \frac{1}{2} \\ -\frac{1}{2} \end{cases}$$

The number of equiangular lines in $\mathbb{R}^d$ is bounded by the absolute bound:

$$\frac{d(d+1)}{2}.$$

This bound is known to be achieved for $d = 2, 3, 7, 23$, and achievability is unknown in general for large $d$.
Delsarte–Goethals–Seidel (1977), Makhnev (2003), Bannai–M.–Venkov (2004), Nebe–Venkov (2013).

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A graph $G$ in the unique two-graph on $276$ vertices is given in Godsil–Royle, Section 11.8. Its adjacency matrix $A$ has the smallest eigenvalue

$$-3 \text{ with multiplicity } 252.$$

so $\exists X \in \mathbb{R}^{276\times 24}$ with

$$XX^\top = A + 3I.$$

Let

$$V = \{\text{row vectors of } X\}.$$

Then $u \sim v \iff (u, v) = 1$,
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Recall: switching class $\cong \{2^{276}$ choices for $\pm\}$, it contains the graph $K_1 \cup \Gamma$, where $\Gamma$ is the McLaughlin graph

$$SRG(275, 162, 105, 81),$$

and a large number of (Haemers–Tonchev 1996; Nozaki, 2009) $SRG(276, 135, 78, 54)$.

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$H$ "contains" every graph in the switching class, since

$$V \cup \{r - x : x \in V\} \subseteq H.$$

Let $L$ be the lattice generated by $V \cup \{r\}$. $L$ is a discrete subgroup of $\mathbb{R}^{24}$, and a free $\mathbb{Z}$-module of rank $24$.

$$\{x \in L : (x, x) = 2\} = \{\pm r\},$$
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## Theorem (Koolen–M.)

*For a proper sublattice $L' \subsetneq L$, TFAE:*

(1) $\Gamma' = L' \cap H$ *is a connected graph in the switching class (hence $|L' \cap H| = 276$),*

(2) $r \notin L$, $|L : L'| = 2$.

*In this case, $\Gamma'$ is one of the four graphs corresponding to three maximal subgroups*

$$L_3(4) : D_{12},\ M_{23},\ 3^5 : (2 \times M_{11}),$$

*and a non-maximal subgroup $U_3(5) : 2$, of $Co.3$.*

The latter statement is verified by computer by examining the orbit of $Co._3$ on $L/2L$.

None of the four graphs is (strongly) regular.