The regular two-graph on 276 vertices revisited

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(joint work with Jack Koolen)

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- Goethals–Seidel (1975) "The regular two-graph on 276 vertices" established the uniqueness (up to complement)
- Two-graph = Switching class of graphs
- The regular two-graph on 276 = the switching class of $K_1 \cup McL$, where $McL = SRG(275, 162, 105, 81)$ is the McLaughlin graph.
- $McL \leq Co._3$.

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Switching of $\Gamma = (V, E)$ with respect to $U \subseteq V$ is $\Gamma^U = (V, E^U)$, where

$$E^U = \{\{x,y\} \in E : x, y \in U\} \\ \cup \{\{x,y\} \in E : x, y \in V \setminus U\} \\ \cup \{\{x,y\} \notin E : x \in U,\ y \in V \setminus U\}.$$

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$\Gamma = L(K_8)$: line graph of K_8, can be defined as

$$\begin{aligned} V &= V(L(K_8)) \\ &= \{e_i + e_j : 1 \le i < j \le 8\} \\ &\subseteq D_8 \subseteq \mathbb{R}^8. \end{aligned}$$

For $u, v \in V$,

$$u \sim v \iff (u, v) = 1.$$

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$$\begin{aligned} r &= \frac{1}{2}(1,1,1,1,1,1,1,1), \\ H &= \{x \in \mathbb{R}^8 : (r, x) = 1\}. \end{aligned}$$

Then

$$\begin{aligned} (r, r) &= 2, \\ V &= \{e_i + e_j : 1 \le i < j \le 8\} \subseteq H. \end{aligned}$$

In fact, $V \cup \{r\}$ is a part of the E_8 root system,

$$H \cap E_8 = V \cup \{r - u : u \in V\}.$$

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Write

$$\overline{u} = u - \frac{1}{2}r \quad (u \in V).$$
$$\{\pm\overline{u} : u \in V\}$$

gives a set of 28 equiangular lines in $H \cong \mathbb{R}^7$. The number of equiangular lines in $\mathbb{R}^d$ is bounded by the absolute bound (Gerzon bound):

$$\frac{d(d+1)}{2}.$$

This bound is known to be achieved for $d = 2, 3, 7, 23$, and achievability is unknown in general for large d.
Delsarte–Goethals–Seidel (1977), Makhnev (2003), Bannai–M.–Venkov (2004), Nebe–Venkov (2013).

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$E_8 \supseteq$ the switching class $\ni K_1 \cup Sch$
(in fact, $E_8 \cap H$)
$D_8 \supseteq L(K_8)$

$??\supseteq$ the switching class $\ni K_1 \cup McL$
$??\supseteq ??,$ hyperplane?

Indeed, there exists an integral lattice L and its affine hyperplane H such that

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Moreover, L contains a unique root (up to ± 1) such that

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Let

$$L_3 = \{x \in L : (x, x) = 3\},$$

and let $V \subseteq L_3 \cap H$ be the 276-element subset representing an arbitrary member of the switching class of $K_1 \cup McL$. Then

$$L_3 \cap H = V \cup \{r - x : x \in V\},$$

$$L_3 = (L \cap H) \cup (-(L \cap H))$$

An analogue of $D_8 \supseteq L(K_8)$? SRG? The switching class is known to contain a large number of $SRG(276, 135, 78, 54)$ (Haemers–Tonchev 1996; Nozaki, 2009).

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Theorem (Koolen–M.)

For a proper sublattice $L' \subsetneq L$, TFAE:

(1) $\Gamma' = L'_3 \cap H$ *is a connected graph in the switching class (hence $|L'_3 \cap H| = 276$),*

(2) $r \notin L$, $|L : L'| = 2$.

In this case, Γ' is one of the four graphs corresponding to three maximal subgroups

$$3^5 : (2 \times M_{11}) \text{ (Goethals–Seidel 1975)},$$
$$M_{23} \text{ (Godsil–Royle 2001)},$$
$$L_3(4) : D_{12} \ (3 + 105 + 168),$$

and a non-maximal subgroup $U_3(5) : 2$ $(1 + 100 + 175)$ of $Co.3$.

None of the four graphs is (strongly) regular.