Maximality of Seidel matrices and switching roots of graphs

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The regular two-graph on 276 vertices revisited

(joint work with Jack Koolen)

Contents

$L(K_8)$ denotes the line graph of the complete graph K_8. Also known as the triangular graph T_8, or Johnson graph $J(8,2)$.

$$\mathsf{D}_n = \{x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \in 2\mathbb{Z}\}.$$

The set of roots

$$R(\mathsf{D}_n) = \{\text{permutations of } ((\pm 1)^2\, 0^{n-2})\}.$$

contains

$$\{\text{permutations of } (1^2\, 0^{n-2})\}$$

which can be regarded as the vertex set of $L(K_n)$.

Equiangular lines and the absolute bound

- Goethals–Seidel (1975) "The regular two-graph on 276 vertices" established the uniqueness (up to complement)
- Two-graph = Switching class of graphs
- The regular two-graph on 276 = the switching class of $McL \cup K_1$, where $McL = SRG(275, 162, 105, 81)$ is the McLaughlin graph.
- $Co._3 \geq McL : 2$, index$= 276$.

The number of equiangular lines in $\mathbb{R}^d$ is bounded by the absolute bound (Gerzon bound):

$$\frac{d(d+1)}{2}.$$

This bound is known to be achieved for $d = 2, 3, 7, 23$, and achievability is unknown in general for large d.

Some d were ruled out by Delsarte–Goethals–Seidel (1977), Makhnev (2003), Bannai–M.–Venkov (2004), Nebe–Venkov (2013).

Let $\Gamma = (V, E)$ be a graph. Switching of Γ with respect to $U \subseteq V$ is $\Gamma^U = (V, E^U)$, where

$$\begin{aligned} E^U = &\{\{x, y\} \in E : x, y \in U\} \\ &\cup \{\{x, y\} \in E : x, y \in V \setminus U\} \\ &\cup \{\{x, y\} \notin E : x \in U,\ y \in V \setminus U\}. \end{aligned}$$

The switching class of Γ is

$$\{\Gamma^U : U \subseteq V\}.$$

The Seidel matrix $S(\Gamma)$ of Γ is

$$S(\Gamma) = J - I - 2A(\Gamma),$$

where $A(\Gamma)$ is the adjacency matrix. Then switching corresponds to the operation

$$S(\Gamma) \mapsto \Delta S(\Gamma)\Delta$$

where Δ is the diagonal matrix with ± 1 on the diagonal.

$L(K_8)$ and $\mathsf{D}_8 \subseteq \mathsf{E}_8$

A representation of norm m of a graph $\Gamma = (V, E)$ means an injective mapping $V \to \mathbb{R}^d$, $x \mapsto u_x$, where

$$(u_x, u_y) = \begin{cases} m & \text{if } x = y, \\ 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Such a representation exists if and only if its Gram matrix $A(\Gamma) + mI$ is positive semidefinite, or equivalently, $\lambda_{\min}(A) \geq -m$.

For $\Gamma = L(K_8)$, $\lambda_{\min}(\Gamma) = -2$. It has a representation of norm 2 as follows:

$$\begin{aligned} V &= V(L(K_8)) \\ &= \{e_i + e_j : 1 \leq i < j \leq 8\} \\ &= \{\text{permutations of } (1^2\, 0^{n-2})\} \\ &\subseteq \mathsf{D}_8 \subseteq \mathbb{R}^8. \end{aligned}$$

For $x, y \in V$,

$$x \sim y \iff (x, y) = 1.$$

$$r = \frac{1}{2}(1,1,1,1,1,1,1,1),$$
$$H = \{x \in \mathbb{R}^8 : (r,x) = 1\}.$$

Then

$$(r,r) = 2,$$
$$V = \{e_i + e_j : 1 \leq i < j \leq 8\} \subseteq H.$$

In fact, $V \cup \{r\}$ is a part of the E_8 root system,

$$H \cap E_8 = V \cup \{r - x : x \in V\}.$$

$$(x,y) = \begin{cases} 1 \\ 0 \end{cases} \iff (x, r-y) = \begin{cases} 0 \\ 1 \end{cases}$$

Root systems and Seidel matrices of largest eigenvalue 3

Write

$$\overline{u} = u - \frac{1}{2}r \quad (u \in V).$$

$$\{\pm\overline{u} : u \in V\}$$

gives a set of 28 equiangular lines in $H \cong \mathbb{R}^7$.

If, for a graph $\Gamma = (V, E)$,

- $\{u_x : x \in V\}$ is a set of vectors of norm 2,
- $(r, u_x) = 1$ for all $x \in V$,
- $(r, r) = 2$,

then replacing u_x by $r - u_x$ corresponds to switching.

We call r a switching root of Γ.

Proposition

Suppose $\lambda_{\min}(\Gamma) \geq -2$. Let $\tilde{\Gamma} = \Gamma * K_1$. TFAE

1. there exists a switching root of Γ
2. $\lambda_{\min}(\tilde{\Gamma}) \geq -2$
3. $\lambda_{\max}(S(\Gamma)) \leq 3$.

$$B(\Gamma) = \begin{bmatrix} A(\Gamma) + 2I & \mathbf{1} \\ \mathbf{1}^\top & 2 \end{bmatrix} = A(\tilde{\Gamma}) + 2I$$

$$\operatorname{rank} B(\Gamma) = \operatorname{rank}(A(\Gamma) + 2I) + 1.$$

Equiangular lines with angle $\arccos 1/3$

$\Longrightarrow$ Seidel matrix S with $3I - S \geq 0$, i.e.,

$\lambda_{\max}(S) \leq 3$

$\Longrightarrow$ Graph Γ with $\lambda_{\min}(\tilde{\Gamma}) \geq -2$, i.e.,

$A(\tilde{\Gamma}) + 2I \geq 0$

$\Longrightarrow$ Graph Γ such that

$\tilde{\Gamma}$ has a representation of norm 2 in a root system.

Weren't they all known in 1970's?

$N(d)$ = max. # equiangular lines in $\mathbb{R}^d$

$N_\alpha(d)$ = max. # equiangular lines in $\mathbb{R}^d$ with angle $\arccos(\alpha)$

$N_\alpha^*(d)$ = max. # equiangular lines in $\mathbb{R}^d$ with angle $\arccos(\alpha)$, rank exactly d

$$N_\alpha(d) = \max_{r \leq d} N_\alpha^*(r).$$

Lemmens–Seidel (1973):

d	4	5	6	7	8	$\cdots$	14
$N(d) = N_{1/3}(d)$	6	10	16	28	28	$\cdots$	28

Glazyrin–Yu (2018):

$$N^*_{1/3}(d) < 28 \quad (8 \le d \le 11).$$

Lin–Yu (2020):

$$N^*_{1/3}(8) = 14 \quad (\text{achieved only by } L(K_{2,7}))$$

Theorem (Cao–Koolen–M.–Yoshino, 2021+)

d	4	5	6	7	≥8
$N^*_{1/3}(d)$	6	10	16	28	$2(d-1)$

The only bound-achieving Seidel matrices $S(\Gamma)$ are

d	4	5	6	7	≥8
Γ	$L(K_{2,3})$	$L(K_5)$	$L(K_6) \cup K_1$	$L(K_8)$	$L(K_{2,d-1})$

$$\mathsf{E}_8 = \mathsf{D}_8 + \frac{1}{2}\mathbb{Z}\mathbf{1},$$
$$\mathsf{E}_7 = \{u \in \mathsf{E}_8 : (u, e_1 - e_2) = 0\},$$
$$\mathsf{E}_6 = \{u \in \mathsf{E}_8 : (u, e_1 - e_2) = (u, e_2 - e_3) = 0\}.$$

Containment relations between root systems is as follows (Cameron–Goethals–Seidel–Shult, 1978):

$$\mathsf{D}_4 \subset \mathsf{D}_5 \subset \cdots,$$
$$\mathsf{E}_6 \subset \mathsf{E}_7 \subset \mathsf{E}_8,$$
$$\mathsf{D}_6 \not\subset \mathsf{E}_6,$$
$$\mathsf{D}_7 \not\subset \mathsf{E}_7,$$
$$\mathsf{D}_8 \subset \mathsf{E}_8,$$
$$\mathsf{E}_n \not\subset \mathsf{D}_{n'} \text{ for } n \text{ and } n'.$$

Let $R = \mathsf{D}_n$ or E_n. Fix $r \in R$. Then

$$N = \{x \in R : (r, x) = 1\}$$

can be regarded as a switching class of a graph. We call this the switching class of R.
Indeed, let $r = (1, 1, 0^{n-2}) \in R(\mathsf{D}_n)$. Then

$$N = \{(1, 0, [(\pm 1)^1, 0^{n-3}])\} \cup \{(0, 1, [(\pm 1)^1, 0^{n-3}])\}$$

represents the switching class of $L(K_{2,n-2})$.

- E_6: $L(K_5)$
- E_7: $L(K_6) \cup K_1$
- E_8: $L(K_8)$

Maximality of Seidel matrices

Recall that a Seidel matrix is a symmetric matrix with zero diagonal, $\pm$ in off-diagonal entries.

If S is a principal submatrix of a Seidel matrix S', then

$$\lambda_{\max}(S) \leq \lambda_{\max}(S'),$$
$$\operatorname{rank}(S) \leq \operatorname{rank}(S').$$

We say that S is maximal if there is no larger Seidel matrix S' satisfying

$$\lambda_{\max}(S) = \lambda_{\max}(S')$$
$$\operatorname{rank}(S) = \operatorname{rank}(S').$$

Lin–Yu (2020) call equiangular lines obtained from maximal Seidel matrices saturated.

We say that S is strongly maximal if there is no larger Seidel matrix S' satisfying

$$\lambda_{\max}(S) = \lambda_{\max}(S')$$

$$\mathsf{D}_4 \subset \mathsf{D}_5 \subset \cdots,$$
$$\mathsf{E}_6 \subset \mathsf{E}_7 \subset \mathsf{E}_8,$$
$$\mathsf{D}_6 \not\subset \mathsf{E}_6,$$
$$\mathsf{D}_7 \not\subset \mathsf{E}_7,$$
$$\mathsf{D}_8 \subset \mathsf{E}_8,$$
$$\mathsf{E}_n \not\subset \mathsf{D}_{n'} \text{ for } n \text{ and } n'.$$

- D_n: $L(K_{2,n-2})$
- E_6: $L(K_5)$
- E_7: $L(K_6) \cup K_1$
- E_8: $L(K_8)$

Theorem

Let $S = S(\Gamma)$, $\lambda_{\max}(S) = 3$, $\operatorname{rank}(3I - S) = d$. Suppose S is maximal.

1. If $d = 5$, then $\Gamma = L(K_5), L(K_{2,4})$,
2. If $d = 6$, then $\Gamma = L(K_6) \cup K_1, L(K_{2,5})$,
3. If $d = 7$, then $\Gamma = L(K_8)$,
4. If $d = 3, 4$ or $r \geq 8$, then $\Gamma = L(K_{2,r-1})$,

up to switching.

If S is strongly maximal, then $\Gamma = L(K_8)$ up to switching.

Results and conjectures

Theorem

A Seidel matrix S of order n achieving the absolute bound

$$n = \frac{d(d+1)}{2},$$

where $d = \mathrm{rank}(\lambda_{\max}(S)I - S)$, is strongly maximal.

Examples: $d = 2, 3, 7, 23$.

$d = 2$, $\lambda = 2$: Unique set of 3 lines with angle $\pi/3$.

$$S = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$d = 3$, $\lambda = \sqrt{5}$: Unique set of 6 lines (the diagonals of the icosahedron). These are the unique strongly maximal Seidel matrices (up to switching) of largest eigenvalue 2 and $\sqrt{5}$.

Classification of root systems is essential in proving the uniqueness of strongly maximal Seidel matrix with $\lambda_{\max} = 3$, but no similar tools are available for $\lambda_{\max} = 5$ (McL).

For n odd, $\overline{K_n}$ is strongly maximal, with $\lambda_{\max} = n - 1$.

$$B_\theta(\Gamma) = \begin{bmatrix} A(\Gamma) + \theta I & \mathbf{1} \\ \mathbf{1}^\top & 2 \end{bmatrix}$$

Theorem

TFAE:

1. $B_\theta(\Gamma) \geq 0$
2. $\lambda_{\max} S(\Gamma) \leq 2\theta - 1.$

If $\lambda_{\max}(S(\Gamma)) = 5$, for example, $\Gamma = McL \cup K_1$, then

$$\begin{bmatrix} A(\Gamma) + 3I & \mathbf{1} \\ \mathbf{1}^\top & 2 \end{bmatrix}$$

Γ has a representation of norm 3 in $\mathbb{R}^{24}$ contained in an affine hyperplane

$$H = \{x \in \mathbb{R}^{24} : (r, x) = 1\},$$

where $(r, r) = 2$.
The lattice generated by the image of Γ admits $Co.3$ as automorphism group.

As an analogue to the case $\lambda_{\max}(S) = 3$, we ask:

Problem

Is $McL \cup K_1$ the only strongly maximal Seidel matrix with largest eigenvalue 5, up to switching?