Neighbor-balanced bijections of hypercubes

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Balanced set

$\mathbf{F}_2 = \{0, 1\}$. The n-dimensional hypercube (also known as the Hamming graph $H(n, 2)$) is the graph

- vertex set $\mathbf{F}_2^n$
- edge set $\{\{u, u + e_j\} \mid u \in \mathbf{F}_2^n,\ j \in [n]\}$

where $e_1, \dots, e_n$ denote the standard basis of $\mathbf{F}_2^n$.

$n = 2$. Neighbors are

$$N(00) = N(11) = \{01, 10\}$$
$$N(01) = N(10) = \{00, 11\}$$

A subset $X \subset \mathbf{F}_2^n$ is called balanced (also known as an orthogonal array of strength 1) if

$$|\{x \in X \mid x_j = 1\}| = \frac{|X|}{2} \quad (\forall j \in [n]).$$

Regarding X as an $|X| \times n$ matrix, this means that every column of X has the same number of 0 and 1, or equivalently, every column of X has weight $|X|/2$.

$$X = \begin{bmatrix} (\qquad\qquad\qquad) \\ (\qquad\qquad\qquad) \\ \vdots \\ (\qquad\qquad\qquad) \end{bmatrix} \qquad N(00) = \begin{bmatrix} (01) \\ (10) \end{bmatrix} \qquad N(01) = \begin{bmatrix} (00) \\ (11) \end{bmatrix}$$

For $a \in \mathbf{F}_2^n$, the weight $\mathrm{wt}(a)$ of a is

$$\mathrm{wt}(a) = |\{j \in [n] \mid a_j = 1\}|.$$

Neighbor-balanced bijections

A bijection $f : \mathbf{F}_2^n \to \mathbf{F}_2^n$ is called neighbor-balanced if

$$f(N(u)) \text{ is balanced } (\forall u \in \mathbf{F}_2^n).$$

$n = 2$, $\mathrm{id}_{\mathbf{F}_2^2}$ is neighbor-balanced.
For n odd, $|N(u)| = n$, $N(u)$ or its image $f(N(u))$ is never balanced.

The following construction is due to Gregor–Kovář (2013): Suppose $n = 4p + 2$. Define a linear transformation $f\colon \mathbf{F}_2^n \to \mathbf{F}_2^n$ by $f(u) = Mu$, where

$$M = \begin{bmatrix} 1 & 0 & \mathbf{1}_{2p} & 0 \\ 0 & 1 & 0 & \mathbf{1}_{2p} \\ 0 & 0 & I_{2p} & J_{2p} \\ 0 & 0 & J_{2p} & I_{2p} \end{bmatrix} \in GL(n, 2)$$

Then $f(N(0)) = \{Me_j \mid j \in [n]\}$ consists of the row vectors of

$$M^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mathbf{1}_{2p}^\top & 0 & I_{2p} & J_{2p} \\ 0 & \mathbf{1}_{2p}^\top & J_{2p} & I_{2p} \end{bmatrix}$$

in which every column has $2p + 1$ zeros and ones.

Since

$$f(N(u)) = f(N(0) + u) = f(N(0)) + f(u)$$

is also balanced, f is neighbor-balanced.

Proposition (Gregor–Kovář, 2013)

If every row of $M \in GL(n,2)$ has weight $n/2$, then the linear transformation defined by M is neighbor-balanced.

If $f\colon \mathbf{F}_2^n \to \mathbf{F}_2^n$ is neighbor-balanced, then so is $\alpha \circ f \circ \beta$, for $\alpha, \beta \in \operatorname{Aut} H(n,2)$.

This means that the set of neighbor-balanced bijections is a union of double cosets of $\operatorname{Aut} H(n,2)$ in the full symmetric group $\operatorname{Sym} \mathbf{F}_2^n$.

Note

$$\operatorname{Aut} H(n,2) = \mathbf{F}_2^n \rtimes \operatorname{Sym}(n) \le AGL(n,2) \le \operatorname{Sym} \mathbf{F}_2^n,$$

where $AGL(n,2) = \mathbf{F}_2^n \rtimes GL(n,2)$.

1. We characterize neighbor-balanced bijections in $AGL(n,2)$ in terms of eigenvectors of the adjacency matrix of $H(n,2)$.
2. We disprove a conjecture of Gregor–Kovář (2013):

$$\{\text{neighbor-balanced bijections}\} \subseteq AGL(n,2)$$

   by constructing a nonlinear neighbor-balanced bijection g with $g(0) = 0$.

A bijection $\gamma\colon \mathbf{F}_2^n \to \{0,1,\ldots,2^n-1\}$ is a distance magic labeling if

$$\sum_{x\in N(u)} \gamma(x) \text{ is a constant } \frac{n(2^n-1)}{2}$$

Regarding $\gamma \in \mathbb{R}^{2^n}$ and using the adjacency matrix A of the hypercube, this means

$$A\gamma = \text{constant}\cdot \mathbf{1} = \frac{n(2^n-1)}{2}$$

Distance magic labelings of graphs were studied by Stanley (1973).

Proposition (Gregor–Kovář, 2013)

Let $\zeta\colon \mathbf{F}_2^n \to \{0,1\ldots,2^n-1\}$ be the inverse of the 2-adic expansion. If $f: \mathbf{F}_2^n \to \mathbf{F}_2^n$ is neighbor-balanced, then $\zeta \circ f$ is a distance magic labeling.

$$\sum_{x\in N(u)} \zeta\circ f(x) = \frac{n(1+2+\cdots+2^{n-1})}{2} = \frac{n(2^n-1)}{2}.$$

coordinates

$$f(N(u)) \begin{cases} (0 \quad\quad\quad\quad) \xrightarrow{\zeta} \{0,1,\ldots,2^n-1\} \\ \vdots \\ (0 \quad\quad\quad\quad) \\ (1 \quad\quad\quad\quad) \\ \vdots \\ (1 \quad\quad\quad\quad) \end{cases}$$

Spectrum of the hypercube

If $\gamma\colon \mathbf{F}_2^n \to \{0,1,\ldots,2^n-1\}$ is a distance magic labeling, then

$$A\gamma = \frac{n(2^n-1)}{2}\cdot\mathbf{1} = \frac{2^n-1}{2}A\mathbf{1}.$$

So

$$\gamma - \frac{2^n-1}{2}\mathbf{1} \in \operatorname{Ker} A.$$

Spectrum of A consists of $n-2i$ $(i=0,1,\ldots,n)$.

$$V_i = \operatorname{Ker}(A-(n-2i)I) \quad (i=0,1,\ldots,n),$$
$$V_{n/2} = \operatorname{Ker} A.$$

Define

$$\chi(a) = ((-1)^{\langle a,x\rangle})_{x\in\mathbf{F}_2^n} \in \mathbb{R}^{2^n}.$$

Then

$$A\chi(a) = (n-2\,\mathrm{wt}(a))\chi(a).$$

So V_i has basis $\{\chi(a) \mid a\in\mathbf{F}_2^n,\ \mathrm{wt}(a)=i\}$.

In particular, $\dim\operatorname{Ker} A = \binom{n}{n/2}$, and $\operatorname{Ker} A$ has basis

$$\{\chi(a) \mid a\in\mathbf{F}_2^n,\ \mathrm{wt}(a)=n/2\}.$$

Recall that $\zeta\colon \mathbf{F}_2^n \to \{0, 1 \ldots, 2^n - 1\}$ is the inverse of the 2-adic expansion.

If $f\colon \mathbf{F}_2^n \to \mathbf{F}_2^n$, $f(u) = Mu$ is neighbor balanced, then $\zeta \circ f$ is a distance magic labeling, so

$$\zeta \circ f - \frac{2^n - 1}{2}\mathbf{1} \in \operatorname{Ker} A$$

It can be shown:

$$\zeta \circ f - \frac{2^n - 1}{2}\mathbf{1} = -\sum_{i \in [n]} 2^{i-2}\chi(M_i),$$

where M_i denotes the i-th row vector of M (and hence $\operatorname{wt}(M_i) = n/2$).

Theorem (M.–Steven S. Tanujaya)

Let $f\colon \mathbf{F}_2^n \to \mathbf{F}_2^n$ be neighbor balanced. Then $f \in AGL(n, 2)$ if and only if

$$\exists M_1, \ldots, M_n \in \{x \in \mathbf{F}_2^n \mid \operatorname{wt}(x) = n/2\},$$
$$\exists \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$$

such that

$$\zeta \circ f - \frac{2^n - 1}{2}\mathbf{1} = \sum_{i \in [n]} \epsilon_i 2^{i-2}\chi(M_i).$$

We next construct neighbor-balanced bijections $g \notin AGL(n, 2)$.

The case $n \equiv 0 \pmod 4$

Proposition

If $n \equiv 0 \pmod 4$, then there is no distance magic labeling for $H(n,2)$.

According to Gregor and Kovář (2013), this is due to Barrientos–Cichacz–Fronček–Krop–Raridan.

So there is no neighbor-balanced bijection for $H(n,2)$ if $n \equiv 0 \pmod 4$.

This proposition can be proved by observing the action of distance-n matrix A_n on $\operatorname{Ker} A$.

If n is even and $\operatorname{wt}(a) = n/2$, then

$$\begin{aligned} A_n\chi(a) &= (-1)^{n/2}\chi(a) \\ &= \chi(a) \qquad\qquad \text{if } n \equiv 0 \pmod 4. \end{aligned}$$

This means that the eigenvalue of A_n on $\operatorname{Ker} A$ is 1, so A_n fixes the vector

$$\gamma - \frac{2^n - 1}{2}\mathbf{1}$$

consisting of 2^n distinct entries. This is impossible since A_n is a permutation matrix of order 2.

Nonlinear neighbor-balanced bijections

Now assume $n \geq 6$ and $n \equiv 2 \pmod 4$.

Gregor–Kovář (2013) conjectured:

Conjecture

Every neighbor-balanced bijection of $\mathbf{F}_2^n$ is affine linear, that is, an element of $AGL(n, 2)$.

We present counterexamples.

Let $n = 4p + 2$. Define $f \colon \mathbf{F}_2^n \to \mathbf{F}_2^n$ by $f(u) = Mu$, where

$$M = \begin{bmatrix} 1 & 0 & \mathbf{1}_{2p} & 0 \\ 0 & 1 & 0 & \mathbf{1}_{2p} \\ 0 & 0 & I_{2p} & J_{2p} \\ 0 & 0 & J_{2p} & I_{2p} \end{bmatrix}$$

Then f is a neighbor-balanced bijection,

$$f \in GL(n, 2) \leq AGL(n, 2).$$

We modify f slightly to produce a nonlinear neighbor-balanced bijection g with $g(0) = 0$.

A construction

Define $\sigma\colon \mathbf{F}_2^3 \to \mathbf{F}_2^3$ by

$$\sigma(u) = \begin{cases} 001 & \text{if } u = 110, \\ 110 & \text{if } u = 001, \\ u & \text{otherwise.} \end{cases}$$

Note that σ is not linear, since

$$\sigma(e_1 + e_2) = e_3 \neq e_1 + e_2 = \sigma(e_1) + \sigma(e_2).$$

Define $g\colon \mathbf{F}_2^n \to \mathbf{F}_2^n$ by

$$g = (\sigma \times \mathrm{id}_{\mathbf{F}_2^{n-3}}) \circ f$$

$$g\colon \mathbf{F}_2^n \xrightarrow{f} \mathbf{F}_2^n = \begin{matrix} \mathbf{F}_2^3 & \xrightarrow{\sigma} & \mathbf{F}_2^3 \\ \oplus & & \oplus \\ \mathbf{F}_2^{n-3} & \xrightarrow{\mathrm{id}} & \mathbf{F}_2^{n-3} \end{matrix} = \mathbf{F}_2^n$$

We claim that g is a neighbor-balanced bijection.

Clearly, $g(0) = 0$, and g is not linear, that is, $g \notin AGL(n, 2)$.

g is neighbor-balanced

Note $g(N(u)) = \{g(u + e_j) \mid j \in [n]\}$.

To show that g is neighbor-balanced, we need:

$$|\{j \in [n] \mid g(u + e_j)_i = 1\}| = \frac{n}{2} \quad (i \in [n]).$$

Since $g = (\sigma \times \mathrm{id}_{\mathbf{F}_2^{n-3}}) \circ f$,

$$g(u + e_j)_i = f(u + e_j)_i \quad (i \in \{4, 5, \dots, n\}).$$

Since f is neighbor-balanced,

$$\begin{aligned} &|\{j \in [n] \mid g(u + e_j)_i = 1\}| \\ &= |\{j \in [n] \mid f(u + e_j)_i = 1\}| \\ &= \frac{n}{2} \quad (i \in \{4, 5, \dots, n\}). \end{aligned}$$

It remains to check

$$|\{j \in [n] \mid g(u + e_j)_i = 1\}| = \frac{n}{2} \quad (i \in \{1, 2, 3\}).$$

Let

$$E = \{0, e_1, e_2, e_3\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Lemma

Let $v_1, v_2, v_3 \in \mathbf{F}_2^n$, and suppose

$$\mathrm{wt}(v_1) = \mathrm{wt}(v_2) = \mathrm{wt}(v_3) = \mathrm{wt}(v_1 + v_2 + v_3) = \frac{n}{2}.$$

Then for $\forall b \in \mathbf{F}_2^3$, the number of column vectors of the matrix

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

belonging to the set $E + b$ is $n/2$.

For example, in the matrix

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

there are 3 vectors in the set

$$E + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Recall $f\colon \mathbf{F}_2^n \to \mathbf{F}_2^n$ was defined as $f(u) = Mu$, where

$$M = \begin{bmatrix} 1 & 0 & \mathbf{1}_{2p} & 0 \\ 0 & 1 & 0 & \mathbf{1}_{2p} \\ 0 & 0 & I_{2p} & J_{2p} \\ 0 & 0 & J_{2p} & I_{2p} \end{bmatrix}$$

$v_i = i$-th row of M $(i = 1, 2, 3)$, satisfy the hypothesis of Lemma. Indeed,

$$\begin{aligned} v_1 &= (1, 0, \mathbf{1}_{2p}, 0_{2p}), \\ v_2 &= (0, 1, 0_{2p}, \mathbf{1}_{2p}), \\ v_3 &= (0, 0, 1, 0_{2p-1}, \mathbf{1}_{2p}), \\ v_1 + v_2 + v_3 &= (1, 1, 0, \mathbf{1}_{2p-1}, 0_{2p}). \end{aligned}$$

all have weight $2p + 1 = n/2$.

Let

$$M' = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \text{the first 3 rows of } M.$$

By Lemma, for $\forall b \in \mathbf{F}_2^3$, the number of column vectors of M' belonigng to the set $E + b$ is $n/2$, i.e.,

$$|\{j \in [n] \mid j\text{-th column of } M' \in E + b\}| = \frac{n}{2}.$$

It remains to check

$$|\{j \in [n] \mid g(u+e_j)_i = 1\}| = \frac{n}{2} \quad (i \in \{1,2,3\}).$$

Recall

$$E = \{000, 100, 010, 001\},$$

$$\sigma(u) = \begin{cases} 001 & \text{if } u = 110, \\ 110 & \text{if } u = 001, \\ u & \text{otherwise,} \end{cases}$$

$$g\colon \mathbf{F}_2^n \xrightarrow{f} \mathbf{F}_2^n = \begin{matrix} \mathbf{F}_2^3 & \xrightarrow{\sigma} & \mathbf{F}_2^3 \\ \oplus & & \oplus \\ \mathbf{F}_2^{n-3} & \xrightarrow{\mathrm{id}} & \mathbf{F}_2^{n-3} \end{matrix} = \mathbf{F}_2^n$$

For $i = 1$, let $\sigma_1\colon \mathbf{F}_2^3 \xrightarrow{\sigma} \mathbf{F}_2^3 \xrightarrow{\pi_1} \mathbf{F}_2$. Then

$$\begin{aligned} & g(u+e_j)_1 = 1 \\ \iff & \sigma_1(f(u+e_j)_1, f(u+e_j)_2, f(u+e_j)_3) = 1 \\ \iff & (f(u+e_j)_i)_{i=1}^3 \in \sigma_1^{-1}(1) \\ \iff & (M(u+e_j)_i)_{i=1}^3 \in \{101, 001, 111, 100\} \\ \iff & M'u + M'e_j \in \{000, 100, 010, 001\} + 101 \\ \iff & j\text{-th column of } M' \in E + (101 + M'u) \end{aligned}$$

Thus

$$|\{j \in [n] \mid g(u+e_j)_1 = 1\}| = \frac{n}{2}.$$

Similar for $i = 2, 3$.

Theorem (M.–Steven S. Tanujaya)

Let $n \geq 6$ with $n \equiv 2 \pmod 4$. Define $\sigma \colon \mathbf{F}_2^3 \to \mathbf{F}_2^3$ by

$$\sigma(u) = \begin{cases} 001 & \text{if } u = 110, \\ 110 & \text{if } u = 001, \\ u & \text{otherwise.} \end{cases}$$

$f \colon \mathbf{F}_2^n \to \mathbf{F}_2^n$ by $f(u) = Mu$, where

$$M = \begin{bmatrix} 1 & 0 & \mathbf{1}_{2p} & 0 \\ 0 & 1 & 0 & \mathbf{1}_{2p} \\ 0 & 0 & I_{2p} & J_{2p} \\ 0 & 0 & J_{2p} & I_{2p} \end{bmatrix},$$

and $g \colon \mathbf{F}_2^n \to \mathbf{F}_2^n$ by

$$g = (\sigma \times \mathrm{id}_{\mathbf{F}_2^{n-3}}) \circ f$$

Then $g \notin AGL(n,2)$ and g is neighbor-balanced.

Gregor–Kovář also posed a problem

Problem

For $n \geq 6$ and $n \equiv 2 \pmod 4$, Find a distance magic labeling of the n-dimensional hypercube that is not of the form $\zeta \circ f$ for any neighbor-balanced bijection f.

Recently, Savický arXiv:2102.08212 presented examples for $n = 6$.