# Association Schemes and Spherical Designs

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# Definition of a Spherical Design

A spherical $t$-design $X$ is a finite subset of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ s.t.

$$\frac{\int_{S^{n-1}} f \, d\mu}{\int_{S^{n-1}} 1 d\mu} = \frac{1}{|X|} \sum_{x \in X} f(x)$$

holds for any polynomial $f(x)$ of degree $\leq t$.

This is useful if one wants to investigate properties of a spherical design, but not convenient if one wants to prove something is a spherical design. . . .

Equivalently, $$\sum_{x,y \in X} Q_j(\langle x, y \rangle) = 0 \qquad (j = 1, 2, \ldots, t),$$

where $\{Q_j\}_{j=0}^{\infty}$ are suitably normalized Gegenbauer polynomials, defined by $Q_0(x) = 1$, $Q_1(x) = nx$,

# Association Scheme

A (symmetric) association scheme is a pair $(X, \{R_i\}_{i=0}^{d})$, where $X$ is a finite set, $R_i$ is a (symmetric) relation on $X \times X$ such that

(i) $R_0$ is the diagonal relation.

(ii) $\{R_i\}_{0 \le i \le d}$ is a partition of $X \times X$.

(iii) For any $i, j, k \in \{0, 1, \ldots, d\}$, the number

$$p_{ij}^k = |\{\gamma \in X \mid (\alpha, \gamma) \in R_i,\ (\gamma, \beta) \in R_j\}|$$

is independent of the choice of $(\alpha, \beta)$ in $R_k$, and $p_{ij}^k = p_{ji}^k$.

For $i \in \{0, \ldots, d\}$, let $A_i$ be the adjacency matrix of the relation $R_i$:

$$(A_i)_{\alpha,\beta} := \begin{cases} 1 & \text{if } (\alpha, \beta) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

# Bose–Mesner Algebra

The linear combinations of the adjacency matrices form a commutative algebra over $\mathbb{R}$ called the Bose–Mesner algebra $\mathfrak{A}$.
Let $E$ be a primitive idempotent of $\mathfrak{A}$, $E \neq \frac{1}{|X|}J$. Then $E$ is a real symmetric positive-semidefinite matrix of rank $n = \operatorname{tr} E$.

$$E = {}^tFF$$

where $F$ is a $n \times |X|$ matrix

$$\frac{|X|}{n}E = {}^tFF \qquad \text{diagonals} = 1$$

where $F$ is a $n \times |X|$ matrix ($x$-th column=$\overline{x}$), and

$$\{\text{column vectors of } F\} = \{\overline{x} \mid x \in X\} \subset S^{n-1} \subset \mathbb{R}^n.$$

If $|X|E = \sum_{i=0}^{d} \theta_i^* A_i$, then

# Spherical Representation

A spherical representation of a symmetric association scheme forms a spherical $t$-design iff

$$\sum_{x,y\in X} Q_j(\langle \overline{x}, \overline{y} \rangle) = 0 \qquad (j = 1, 2, \ldots, t).$$

Equivalently,

$$\sum_{i=0}^{d} k_i Q_j(\frac{\theta_i^*}{n}) = 0 \qquad (j = 1, 2, \ldots, t).$$

where $k_i$ is the valency of the relation $R_i$, i.e.,

$$k_i = \frac{|R_i|}{|X|}.$$

# Spherical Representation

$$\sum_{x,y\in X} k_i Q_j(\frac{\theta_i^*}{n}) = 0 \qquad (j = 1, 2, \ldots, t).$$

$$\sum_{x,y\in X} k_i Q_j(\frac{\theta_i^*}{n}) = 0 \qquad (j = 1, 2)$$

always hold, so a spherical representation $\overline{X}$ of a symmetric association scheme $X$ always give a spherical 2-design.
$\overline{X}$ is a 3-design iff $(E \circ E)E = 0$.

Suppose $X$ is Q-polynomial, i.e., if $\exists v_i^*(x)$: polynomial of degree $i$, such that

$$E_i = \frac{1}{|X|} v_i^*(|X|E) \quad (i = 0, 1, \ldots, d)$$



are all the primitive idempotents of $\mathfrak{A}$

# Q-Polynomial Scheme

$$xv_i^*(x) = c_{i+1}^* v_{i+1}^*(x) + a_i^* v_i^*(x) + b_{i-1}^* v_{i-1}^*(x)$$

**Lemma 1.** Let $\overline{X}$ denote the embedding of a Q-polynomial association scheme $X$ into the unit sphere via the primitive idempotent $E = E_1$.

(i) $\overline{X}$ is a 3-design if and only if $a_1^* = 0$.

(ii) $\overline{X}$ is a 4-design if and only if $a_1^* = 0$ and

$$b_0^* b_1^* c_2^* + 2(b_1^* c_2^* - {b_0^*}^2 + b_0^*) = 0.$$

(iii) $\overline{X}$ is a 5-design if and only if $\overline{X}$ is a 4-design and $a_2^* = 0$.

# $U_{2d}(2)$ Dual Polar Graph

Among the known infinite families of P- and Q-polynomial association schemes, only the following family produces spherical $4$-designs, when embedded into the unit sphere via the primitive idempotent $E = E_1$:

The dual polar graph associated with the unitary group $U_{2d}(2)$.

| | |
|---|---|
| vertices: | maximal totally isotropic subspaces |
| adjacency: | intersect at dimension $d-1$ |

$$n = \operatorname{rank} E_1 = \frac{2^{2d}+2}{3}, \qquad \frac{\theta_j^*}{n} = (-\frac{1}{2})^j.$$

In fact, this gives a spherical $5$-design if $d \geq 3$.

# Strongly Perfect Lattices

A lattice whose minimal vectors form a spherical $5$-design is called strongly perfect.

Up to dimension $\leq 9$, only certain root lattices and their duals are stronlgy perfect.

**Theorem 1 (Nebe–Venkov).** There are exactly two strongly perfect lattices in dimension $10$: Martinet's lattice $K'_{10}$ and its dual $(K'_{10})^*$.

$K'_{10}$ has $270$ vectors of norm $4$.
$(K'_{10})^*$ has $240$ vectors of norm $6$.

These lattices look very special $\rightarrow$ it must be very nice: $\rightarrow$ association scheme?

$$\text{spherical } t\text{-design} \quad \overset{\text{sufficient condition}}{\Longrightarrow} \quad \text{association scheme}$$

# Degree of a Spherical Design

The degree of a finite subset $\Omega \subset S^{n-1}$ is

$$|\{(x,y) \mid x,y \in \Omega,\ x \neq y\}|.$$

**Theorem 2 (Delsarte–Goethals–Seidel).** If $\Omega$ is a spherical $t$-design of degree $s$ and $2s-2 \leq t$, then $\Omega$ carry a structure of an association scheme.

The shortest vectors of $K'_{10}$ have norm $4$, with degree

$s = |\{4, 2, 1, 0, -1, -2, -4\}| \ = 6$, while $t = 5$.

The shortest vectors of $(K'_{10})^*$ have norm $6$, with degree

$s = |\{6, 3, 2, 1, 0, -1, -2, -3, -6\}| \ = 8$, while $t = 5$.

# Molien Series

Let $G$ be a finite irreducible subgroup of the real orthogonal group $O(n, \mathbb{R})$. The Molien series of $G$ is

$$\Phi_G(q) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - q \cdot g)}.$$

**Theorem 3 (Goethals–Seidel, 1979).** Every $G$-orbit on the sphere is a spherical $t$-design iff

$$(1 - q^2)\Phi_G(q) = 1 + \underbrace{0 \cdot q + \cdots + 0 \cdot q^t}\, + a_{t+1} q^{t+1} + \cdots$$

$$\Phi_{\mathrm{Aut}(K'_{10})}(q) = 1 + 2q^6 + 3q^8 + \cdots$$

# $PSp(4,3)$

$$PSp(4,3) \overset{40}{\supset} \text{line stabilizer} \overset{2}{\supset} H$$
$$\downarrow \qquad\qquad \downarrow$$
$$S_4 \supset A_4$$

Then one obtains an commutative (but not symmetric) association scheme $X = PSp(4,3)/H$ on $80$ points  with 2nd eigenmatrix

$$Q = \begin{bmatrix} 1 & 30 & 24 & 15 & 5 & 5 \\ 1 & -30 & 24 & 15 & -5 & -5 \\ 1 & 0 & 4 & -5 & 5/\sqrt{-3} & -5/\sqrt{-3} \\ 1 & 0 & 4 & -5 & -5/\sqrt{-3} & 5/\sqrt{-3} \\ 1 & 10/3 & -8/3 & 5/3 & -5/3 & -5/3 \\ 1 & -10/3 & -8/3 & 5/3 & 5/3 & 5/3 \end{bmatrix}$$

## $80 \times 3 = 240$

The direct product of two association schemes $X$ and $\mathbb{Z}_3$ has its 2nd eigenmatrix the tensor product:

$$Q = \begin{bmatrix} 1 & 30 & 24 & 15 & 5 & 5 \\ 1 & -30 & 24 & 15 & -5 & -5 \\ 1 & 0 & 4 & -5 & 5/\sqrt{-3} & -5/\sqrt{-3} \\ 1 & 0 & 4 & -5 & -5/\sqrt{-3} & 5/\sqrt{-3} \\ 1 & 10/3 & -8/3 & 5/3 & -5/3 & -5/3 \\ 1 & -10/3 & -8/3 & 5/3 & 5/3 & 5/3 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

Fusing complex conjugates. . . .

# $80 \times 3 = 240$

$$
\begin{bmatrix}
1 & 10 & 48 & 30 & 10 & 10 & \cdots \\
1 & 5 & -24 & -15 & -10 & 5 & \cdots \\
1 & 5 & -4 & 5 & 0 & -5 & \cdots \\
1 & 10/3 & -16/3 & 10/3 & 10/3 & 10/3 & \cdots \\
1 & 5/3 & 8/3 & -5/3 & -10/3 & 5/3 & \cdots \\
1 & 0 & 8 & -10 & 0 & 0 & \cdots \\
1 & -5/3 & 8/3 & -5/3 & 10/3 & -5/3 & \cdots \\
1 & -10/3 & -16/3 & 10/3 & -10/3 & -10/3 & \cdots \\
1 & -5 & -4 & 5 & 0 & 5 & \cdots \\
1 & -5 & -24 & -15 & 10 & -5 & \cdots \\
1 & -10 & 48 & 30 & -10 & -10 &
\end{bmatrix}
\quad
\begin{matrix}
\text{valency} \\
1 \\ 2 \\ 24 \\ 27 \\ 54 \\ 24 \\ 54 \\ 27 \\ 24 \\ 2 \\ 1
\end{matrix}
$$

# Cosine Sequence

$$
\begin{bmatrix}
1 & 10 & \cdots \\
1 & 5 & \cdots \\
1 & 5 & \cdots \\
1 & 10/3 & \cdots \\
1 & 5/3 & \cdots \\
1 & 0 & \cdots \\
1 & -5/3 & \cdots \\
1 & -10/3 & \cdots \\
1 & -5 & \cdots \\
1 & -5 & \cdots \\
1 & -10 & \cdots
\end{bmatrix}
\qquad
\begin{matrix}
\text{valency} \\
1 \\
\left.\begin{matrix} 2 \\ 24 \end{matrix}\right\} \\
27 \\
54 \\
24 \\
54 \\
27 \\
\left.\begin{matrix} 24 \\ 2 \end{matrix}\right\} \\
1
\end{matrix}
$$

gives the cosine sequence

$$
\begin{bmatrix}
1 \\
1/2 \\
1/3 \\
1/6 \\
0 \\
-1/6 \\
-1/3 \\
-1/2 \\
-1
\end{bmatrix}
\begin{matrix}
1 \\
26 \\
27 \\
54 \\
24 \\
54 \\
27 \\
26 \\
1
\end{matrix}
$$

# Conclusion

- The set of $240$ shortest vectors of Martinet's lattice $(K'_{10})^*$ can be reconstructed from

$$\left( \begin{array}{l} \text{permutation representation} \\ \text{of degree } 80 \text{ of } PSp(4,3) \end{array} \right) \bigotimes \mathbb{Z}_3.$$

  Can we generalize this construction to obtain more spherical $5$-designs?

- 270 shortest vectors of $K'_{10}$ form an association scheme?
- Nonsymmetric $\otimes$ Nonsymmetric $\overset{\text{fusion}}{\Longrightarrow}$ symmetric?

Thank you for your attention.