Association Schemes and Spherical Designs*

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1 Spherical Designs

A spherical t-design X is a finite subset of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that

$$\frac{\int_{S^{n-1}} f d\mu}{\int_{S^{n-1}} 1 d\mu} = \frac{1}{|X|} \sum_{x \in X} f(x)$$

holds for any polynomial $f(x)$ of degree $\le t$.

This definition is useful if one wants to investigate properties of a spherical design, but not convenient if one wants to prove something is a spherical design. An equivalent condition is:

$$\sum_{x,y \in X} Q_j(\langle x, y \rangle) = 0 \qquad (j = 1, 2, \dots, t), \tag{1}$$

where $\{Q_j\}_{j=0}^{\infty}$ are suitably normalized Gegenbauer polynomials, defined by $Q_0(x) = 1$, $Q_1(x) = nx$,

$$\frac{j+1}{n+2j} Q_{j+1}(x) = x Q_j(x) - \frac{n+j-3}{n+2j-4} Q_{j-1}(x) \quad (j = 1, 2, 3, \dots).$$

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2 Association Schemes

A (symmetric) association scheme is a pair $(X, \{R_i\}_{i=0}^d)$, where X is a finite set, R_i is a (symmetric) relation on $X \times X$ such that

(i) R_0 is the diagonal relation.

(ii) $\{R_i\}_{0 \le i \le d}$ is a partition of $X \times X$.

(iii) For any $i, j, k \in \{0, 1, \ldots, d\}$, the number

$$p_{ij}^k = |\{\gamma \in X \mid (\alpha, \gamma) \in R_i,\ (\gamma, \beta) \in R_j\}|$$

is independent of the choice of (α, β) in R_k, and $p_{ij}^k = p_{ji}^k$.

For $i \in \{0, \ldots, d\}$, let A_i be the adjacency matrix of the relation R_i:

$$(A_i)_{\alpha,\beta} := \begin{cases} 1 & \text{if } (\alpha, \beta) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

The linear combinations of the adjacency matrices of a symmetric association scheme form a commutative algebra $\mathfrak{A}$ over $\mathbb{R}$ called the Bose–Mesner algebra.

Let E be a primitive idempotent of $\mathfrak{A}$ and $E \neq \frac{1}{|X|}J$. Then E is a real symmetric positive-semidefinite matrix of rank $n = \operatorname{tr} E$. The matrix $\frac{|X|}{n}E$ has all the diagonal entries 1, and we may write it as

$$\frac{|X|}{n}E = {}^tFF$$

where F is a $n \times |X|$ matrix (x-th column=$\overline{x}$), and

$$\{\text{column vectors of } F\} = \{\overline{x} \mid x \in X\} \subset S^{n-1} \subset \mathbb{R}^n.$$

If $|X|E = \sum_{i=0}^d \theta_i^* A_i$, then

$$\langle \overline{x}, \overline{y} \rangle = \frac{\theta_i^*}{n} \quad \text{if } (x, y) \in R_i \qquad \text{(cosines of the vectors).}$$

A spherical representation of a symmetric association scheme forms a spherical t-design iff

$$\sum_{x,y \in X} Q_j(\langle \overline{x}, \overline{y} \rangle) = 0 \qquad (j = 1, 2, \ldots, t).$$

Equivalently,

$$\sum_{i=0}^{d} k_i Q_j(\frac{\theta_i^*}{n}) = 0 \qquad (j = 1, 2, \ldots, t).$$

where k_i is the valency of the relation R_i, i.e.,

$$k_i = \frac{|R_i|}{|X|}.$$

Note that

$$\sum_{x,y \in X} k_i Q_j(\frac{\theta_i^*}{n}) = 0 \qquad\qquad (j = 1, 2)$$

always hold, so a spherical representation $\overline{X}$ of a symmetric association scheme X always give a spherical 2-design.

Moreover, $\overline{X}$ is a 3-design iff $(E \circ E)E = 0$.

We formulate conditions in terms of parameters for a spherical representation to become a spherical t-design for $t \geq 4$, only for Q-polynomial association schemes. Suppose X is Q-polynomial, i.e., if $\exists v_i^*(x)$: polynomial of degree i, such that

$$E_i = \frac{1}{|X|} v_i^*(|X|E) \quad (i = 0, 1, \ldots, d)$$

are all the primitive idempotents of $\mathfrak{A}$.

Then

$$x v_i^*(x) = c_{i+1}^* v_{i+1}^*(x) + a_i^* v_i^*(x) + b_{i-1}^* v_{i-1}^*(x).$$

Lemma 1. Let $\overline{X}$ denote the embedding of a Q-polynomial association scheme X into the unit sphere via the primitive idempotent $E = E_1$.

(i) $\overline{X}$ is a 3-design if and only if $a_1^* = 0$.

(ii) $\overline{X}$ is a 4-design if and only if $a_1^* = 0$ and

$$b_0^* b_1^* c_2^* + 2(b_1^* c_2^* - {b_0^*}^2 + b_0^*) = 0.$$

(iii) $\overline{X}$ is a 5-design if and only if $\overline{X}$ is a 4-design and $a_2^* = 0$.

Among the known infinite families of P- and Q-polynomial association schemes, only the following family produces spherical 4-designs, when embedded into the unit sphere via the primitive idempotent $E = E_1$.

The dual polar graph associated with the unitary group $U_{2d}(2)$ is defined by:

vertices:	maximal totally isotropic subspaces
adjacency:	intersect at dimension $d-1$

Then

$$n = \operatorname{rank} E_1 = \frac{2^{2d}+2}{3}, \qquad \frac{\theta_j^*}{n} = (-\frac{1}{2})^j.$$

In fact, this gives a spherical 5-design if $d \geq 3$ ([3]).

3 Martinet's Lattices

A lattice whose shortest vectors form a spherical 5-design is called strongly perfect.

Up to dimension ≤ 9, only certain root lattices and their duals are stronlgy perfect.

Theorem 1 (Nebe–Venkov [4])**.** There are exactly two strongly perfect lattices in dimension 10: Martinet's lattice K'_{10} and its dual $(K'_{10})^*$.

The lattice K'_{10} has 270 shortest vectors of norm 4, while the lattice $(K'_{10})^*$ has 240 shortest vectors of norm 6.

Since these lattices look very special, it must be very nice. Do the set of shortest vectors form association schemes?

There is a sufficient condition for a spherical t-design to carry a structure of an association scheme. We need a definition to state the condition.

The degree of a finite subset $\Omega \subset S^{n-1}$ is

$$|\{(x,y) \mid x, y \in \Omega,\ x \neq y\}|.$$

Theorem 2 (Delsarte–Goethals–Seidel [1])**.** If Ω is a spherical t-design of degree s and $2s-2 \leq t$, then Ω carry a structure of an association scheme.

The shortest vectors of K'_{10} have norm 4, with degree

$$s = |\{2, 1, 0, -1, -2, -4\}| = 6,$$

while $t = 5$. The shortest vectors of $(K'_{10})^*$ have norm 6, with degree

$$s = |\{3, 2, 1, 0, -1, -2, -3, -6\}| = 8,$$

while $t = 5$. Thus, we can apply Theorem 2 in neither case.

In our case, however, there is an easy way to prove a stronger result if we use a computer a little.

Let G be a finite irreducible subgroup of the real orthogonal group $O(n, \mathbb{R})$. The Molien series of G is

$$\Phi_G(q) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - q \cdot g)}.$$

Theorem 3 (Goethals–Seidel [2])**.** Every G-orbit on the sphere is a spherical t-design iff

$$(1 - q^2)\Phi_G(q) = 1 + \underbrace{0 \cdot q + \cdots + 0 \cdot q^t} + a_{t+1}q^{t+1} + \cdots$$

The following MAGMA session constructs the Martinet's lattice, computes the automorphism group and computes the Molien series.

```
Magma V2.11-1     Sat Jul 24 2004 14:19:43    [Seed = 1713821203]
Type ? for help.  Type <Ctrl>-D to quit.
> ld:=LatticeDatabase();
> K12:=Lattice(ld,12,27); // Coxeter-Todd lattice
> sv:=ShortestVectors(K12);
> v1:=Random(sv);
> v2s:={ x : x in sv | (v1,x)^2 eq 4 };
> v2:=Random({ v : v in v2s |
>     #{ x : x in sv | (v1,x) eq 0 and (v,x) eq 0 } eq 135 });
> v1v2p:={ x : x in sv | (v1,x) eq 0 and (v2,x) eq 0 };
> K:=LatticeWithGram(GramMatrix(Dual(sub< K12 | v1v2p >)));
> G:=AutomorphismGroup(K);
> AutL:=sub< GL(10,Rationals()) | Generators(G) >;
> Pt<q>:=PowerSeriesRing(Rationals(),10);
> (1-q^2)*(Pt!MolienSeries(AutL));
1 + 2*q^6 + 3*q^8 + O(q^10)
```

We obtain

$$\Phi_{\mathrm{Aut}(K'_{10})}(q) = 1 + 2q^6 + 3q^8 + \cdots$$

This means that every orbit of the automorphism group of Martinet's lattice K'_{10} is a spherical 5-design. In particular, the set of shortest vectors of the lattice $(K'_{10})^*$ is a spherical 5-design.

We now give an interpretation of the set of shortest vectors of the lattice $(K'_{10})^*$ in terms of an association scheme. There is a subgroup of index 80 in the projective symplectic group $\mathrm{PSp}(4,3)$:

$$\begin{array}{ccccc} \mathrm{PSp}(4,3) & \overset{40}{\supset} & \text{line stabilizer} & \overset{2}{\supset} & H \\ & & \downarrow & & \downarrow \\ & & S_4 & \supset & A_4 \end{array}$$

This gives a permutation representation of degree 80 of $\mathrm{PSp}(4,3)$. Then one obtains a commutative (but not symmetric) association scheme $X = \mathrm{PSp}(4,3)/H$ on 80 points with 2nd eigenmatrix

$$Q = \begin{bmatrix} 1 & 30 & 24 & 15 & 5 & 5 \\ 1 & -30 & 24 & 15 & -5 & -5 \\ 1 & 0 & 4 & -5 & 5/\sqrt{-3} & -5/\sqrt{-3} \\ 1 & 0 & 4 & -5 & -5/\sqrt{-3} & 5/\sqrt{-3} \\ 1 & 10/3 & -8/3 & 5/3 & -5/3 & -5/3 \\ 1 & -10/3 & -8/3 & 5/3 & 5/3 & 5/3 \end{bmatrix}$$

The direct product of two association schemes X and $\mathbb{Z}_3$ has its 2nd eigenmatrix the tensor product:

$$Q = \begin{bmatrix} 1 & 30 & 24 & 15 & 5 & 5 \\ 1 & -30 & 24 & 15 & -5 & -5 \\ 1 & 0 & 4 & -5 & 5/\sqrt{-3} & -5/\sqrt{-3} \\ 1 & 0 & 4 & -5 & -5/\sqrt{-3} & 5/\sqrt{-3} \\ 1 & 10/3 & -8/3 & 5/3 & -5/3 & -5/3 \\ 1 & -10/3 & -8/3 & 5/3 & 5/3 & 5/3 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

Fusing complex conjugates, we obtain

$$
Q = \begin{bmatrix}
1 & 10 & 48 & 30 & 10 & 10 & \cdots \\
1 & 5 & -24 & -15 & -10 & 5 & \cdots \\
1 & 5 & -4 & 5 & 0 & -5 & \cdots \\
1 & 10/3 & -16/3 & 10/3 & 10/3 & 10/3 & \cdots \\
1 & 5/3 & 8/3 & -5/3 & -10/3 & 5/3 & \cdots \\
1 & 0 & 8 & -10 & 0 & 0 & \cdots \\
1 & -5/3 & 8/3 & -5/3 & 10/3 & -5/3 & \cdots \\
1 & -10/3 & -16/3 & 10/3 & -10/3 & -10/3 & \cdots \\
1 & -5 & -4 & 5 & 0 & 5 & \cdots \\
1 & -5 & -24 & -15 & 10 & -5 & \cdots \\
1 & -10 & 48 & 30 & -10 & -10 &
\end{bmatrix}
\quad
\begin{matrix}
\text{valency} \\ 1 \\ 2 \\ 24 \\ 27 \\ 54 \\ 24 \\ 54 \\ 27 \\ 24 \\ 2 \\ 1
\end{matrix}
$$

This gives rise to a spherical representation:

$$
\begin{bmatrix}
1 & 10 & \cdots \\
1 & 5 & \cdots \\
1 & 5 & \cdots \\
1 & 10/3 & \cdots \\
1 & 5/3 & \cdots \\
1 & 0 & \cdots \\
1 & -5/3 & \cdots \\
1 & -10/3 & \cdots \\
1 & -5 & \cdots \\
1 & -5 & \cdots \\
1 & -10 & \cdots
\end{bmatrix}
\quad
\begin{matrix}
\text{valency} \\ 1 \\ \left.\begin{matrix} 2 \\ 24 \end{matrix}\right\} \\ 27 \\ 54 \\ 24 \\ 54 \\ 27 \\ \left.\begin{matrix} 24 \\ 2 \end{matrix}\right\} \\ 1
\end{matrix}
\quad
\begin{matrix} \text{gives the} \\ \text{cosine sequence} \end{matrix}
\quad
\begin{bmatrix}
1 \\ 1/2 \\ 1/3 \\ 1/6 \\ 0 \\ -1/6 \\ -1/3 \\ -1/2 \\ -1
\end{bmatrix}
\begin{matrix}
1 \\ 26 \\ 27 \\ 54 \\ 24 \\ 54 \\ 27 \\ 26 \\ 1
\end{matrix}
$$

This spherical representation realizes the set of 240 shortest vectors of the lattice $(K'_{10})^*$. One can check that this set forms a spherical 5-design using the definition of the spherical design in terms of the Gegenbauer polynomials (1).

4 Conclusion

- The set of 240 shortest vectors of Martinet's lattice $(K'_{10})^*$ can be reconstructed from

$$\begin{pmatrix} \text{permutation representation} \\ \text{of degree 80 of } \mathrm{PSp}(4,3) \end{pmatrix} \bigotimes \mathbb{Z}_3.$$

  Can we generalize this construction to obtain more spherical 5-designs? It seems important to notice the following aspect of this construction:

$$\text{nonsymmetric} \otimes \text{nonsymmetric} \overset{\text{fusion}}{\Longrightarrow} \text{symmetric}$$

  If one were to fuse pairs of nonsymmetric relations before taking the direct product, one only finds a spherical representation of dimension 20, not 10.

- A more straightforward construction is as follows. Let F be the matrix ${}^tF\overline{F} = \frac{|X|}{n}E$, where E is the primitive idempotent. The Gram matrix of the set $X \cup \omega X \cup \omega^2 X \subset \mathbb{C}^5$ regarded as vectors of $\mathbb{R}^{10}$ is

$$\begin{aligned} \mathrm{Re} \begin{pmatrix} {}^tF \\ \omega{}^tF \\ \omega^2{}^tF \end{pmatrix} \begin{pmatrix} \overline{F} & \overline{\omega F} & \overline{\omega^2 F} \end{pmatrix} &= \mathrm{Re}\, {}^tF\overline{F} \otimes W \\ &= E \otimes W + \overline{E \otimes W}, \end{aligned}$$

  where

$$W = \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}.$$

  Note that the matrix E above gives an embedding of X into a lattice of rank 5 over $\mathbb{Z}[\omega]$.

- Can the set of 270 shortest vectors of K'_{10} be constructed in a similar manner as above?

References

[1] P. Delsarte, J.-M. Goethals and J. J. Seidel, Spherical codes and designs, Geometriae Dedicata 6 (1977), 363–388.

[2] J.-M. Goethals and J. J. Seidel, Spherical designs, Proc. Sympos. Pure Math., XXXIV, 255–272, Amer. Math. Soc., Providence, R.I., 1979.

[3] A. Munemasa, Spherical 5-designs obtained from finite unitary groups, European J. Combin., 25 (2004), 261–267.

[4] G. Nebe and B. Venkov, The strongly perfect lattices of dimension 10, J. Théor. Nombres Bordeaux 12 (2000), 503–518.