The Terwilliger Algebras of Group Association Schemes

Eiichi Bannai
Akihiro Munemasa

The Terwilliger algebra of an association scheme was introduced by Paul Terwilliger [7] in order to study P-and Q-polynomial association schemes. The purpose of this paper is to discuss in detail properties of the Terwilliger algebra of the group association scheme of a finite group. We shall give bounds on the dimension of the Terwilliger algebra, and define triple regularity.

Let G be a finite group, $C_0 = \{e\}$, $C_1, \ldots, C_d$ the conjugacy classes of G. For $i = 0, 1, \ldots, d$, define

$$R_i = \{(x, y) | yx^{-1} \in C_i\}.$$

Then $\mathcal{X}(G) = (G, \{R_i\}_{0 \le i \le d})$ becomes a commutative association scheme, and it is called the group association scheme of the finite group G. Define the adjacency matrix A_i of the relation R_i:

$$(A_i)_{x,y} = \begin{cases} 1 & (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

Then there exist nonnegative integers p_{ij}^k such that $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$. This is equivalent to the relation $\underline{C_i}\,\underline{C_j} = \sum_{k=0}^d p_{ij}^k \underline{C_k}$ where $\underline{C_i} = \sum_{x \in C_i} x \in \mathbf{C}G$ and the multiplication is performed as elements of the group algebra $\mathbf{C}G$. If we put $\mathcal{A} = \langle A_0, \ldots, A_d \rangle_{\mathbf{C}}$, then $\mathcal{A}$ becomes a $(d+1)$-dimensional subalgebra of the matrix algebra $M_{|G|}(\mathbf{C})$, and is called the Bose–Mesner algebra. It is isomorphic to the center of the group algebra. Let $E_0, \ldots, E_d$ be the primitive idempotents of $\mathcal{A}$. Since $\mathcal{A}$ is closed under the Hadamard multiplication, we have $E_i \circ E_j \in \mathcal{A}$, so that there exist complex numbers q_{ij}^k such that

$$E_i \circ E_j = \frac{1}{|G|} \sum_{k=0}^d q_{ij}^k E_k$$

(it is known that q_{ij}^k are indeed real and nonnegative, see [2]).

We define the diagonal matrices E_i^*, A_i^* by

$$(E_i^*)_{xx} = \begin{cases} 1 & x \in C_i \\ 0 & \text{otherwise} \end{cases}$$

$$(A_i^*)_{xx} = |G|(E_i)_{e,x}$$

Then we have

$$A_i^* A_j^* = \sum_{k=0}^{d} q_{ij}^k A_k^*$$

and the algebra

$$\mathcal{A}^* = \langle A_0^*, \ldots, A_d^* \rangle_{\mathbf{C}} = \langle E_0^*, \ldots, E_d^* \rangle_{\mathbf{C}}$$

is called the dual Bose–Mesner algebra.

Now the Terwilliger algebra T is the subalgebra of $M_{|G|}(\mathbf{C})$ generated by $\mathcal{A}$ and $\mathcal{A}^*$. Since T is closed under the conjugate-transpose, T is semisimple, and one can easily see that T is non-commutative if $G \neq 1$.

1 Bounds on $\dim T$

In this section we give upper and lower bounds on the dimension of the Terwilliger algebra of a group association scheme. The following identities will be used in the proof of the next lemma. See [2] for a proof.

$$\bar{E}_i = E_i^T = E_{\hat{\imath}} \quad \text{for some } \hat{\imath} \in \{0, 1, \ldots, d\},$$

$$\operatorname{rank} E_j = |G|(E_j)_{e,e},$$

$$q_{\hat{\imath}k}^j \operatorname{rank} E_j = q_{ij}^k \operatorname{rank} E_k.$$

Lemma 1 (i) $\operatorname{tr}(E_i^* A_j E_k^* \overline{(E_l^* A_m E_n^*)}^T) = \delta_{il}\delta_{jm}\delta_{kn} p_{ij}^k |C_k|$.

(ii) $\operatorname{tr}(E_i A_j^* E_k \overline{(E_l A_m^* E_n)}^T) = \delta_{il}\delta_{jm}\delta_{kn} q_{ij}^k \operatorname{rank} E_k$

Proof. (i) This follows directly from the definition.

(ii) By the identities mentioned above,

$$\begin{aligned}
\operatorname{tr}(E_i A_j^* E_k \overline{(E_l A_m^* E_n)}^T) &= \operatorname{tr}(E_l E_i A_j^* E_k E_n \overline{A_m^*}) \\
&= \delta_{il}\delta_{kn} \operatorname{tr}(E_i A_j^* E_k \overline{A_m^*}) \\
&= \delta_{il}\delta_{kn} |G|^2 \sum_{x,y \in G} (E_i)_{x,y} (E_j)_{e,y} (E_k)_{y,x} (E_m)_{x,e} \\
&= \delta_{il}\delta_{kn} |G|^2 (E_j (E_i \circ E_k^T) E_m)_{e,e} \\
&= \delta_{il}\delta_{kn}\delta_{jm} |G| q_{i\hat{k}}^j (E_j)_{e,e} \\
&= \delta_{il}\delta_{kn}\delta_{jm} q_{ij}^k \operatorname{rank} E_k.
\end{aligned}$$

This completes the proof. $\square$

Consider the subspaces T_0 and T_0^* defined by

$$T_0 = \operatorname{span}_{\mathbf{C}} \{ E_i^* A_j E_k^* | 0 \leq i, j, k \leq d \},$$

$$T_0^* = \mathrm{span}_{\mathbf{C}}\{E_i A_j^* E_k | 0 \le i, j, k \le d\}.$$

By Lemma 1 we have

$$\dim T_0 = |\{(i,j,k) | p_{ij}^k \neq 0\}|,$$

$$\dim T_0^* = |\{(i,j,k) | q_{ij}^k \neq 0\}|.$$

In other words, $\dim T_0$ is the number of triples (i,j,k) such that $(C_i C_j) \cap C_k \neq \emptyset$. There is a one-to-one correspondence between the set of primitive idempotents $\{E_i\}_{0 \le i \le d}$ and the set of complex irreducible characters of G, say $E_i \leftrightarrow \chi_i$. As shown in [2], $q_{ij}^k = \frac{\chi_i(1)\chi_j(1)}{\chi_k(1)}(\chi_i \chi_j, \chi_k)$ holds. Thus, $\dim T_0^*$ is the number of triples (i,j,k) such that $(\chi_i \chi_j, \chi_k) \neq 0$.

Let $\tilde{T} = \mathrm{End}_G(\mathbf{C}G)$ be the centralizer algebra of the permutation representation of G acting on G itself by conjugation. The dimension of $\tilde{T}$ is the number of orbits of G acting on $G \times G$ by simultaneous conjugation. As is well-known, this is equal to the average of fixed points, i.e.,

$$\dim \tilde{T} = \frac{1}{|G|} \sum_{a \in G} |C_G(a)|^2 = \sum_{i=0}^{d} \frac{|G|}{|C_i|}.$$

Theorem 2 *We have the following bounds on the dimension of the Terwilliger algebra T.*

(i) $|\{(i,j,k) | p_{ij}^k \neq 0\}| \le \dim T$.

(ii) $|\{(i,j,k) | q_{ij}^k \neq 0\}| \le \dim T$.

(iii) $\dim T \le \sum_{i=0}^{d} |G|/|C_i|$.

Proof. These are direct consequences of $T_0 \subset T$, $T_0^* \subset T$ and $T \subset \tilde{T}$. $\square$

If G is abelian, then Theorem 2 implies $\dim T_0 = |G|^2$, i.e., T coincides with the full matrix algebra $M_{|G|}(\mathbf{C})$.

2 Triple regularity

If the finite group G acts transitively on the set

$$S_{ijk} = \{(g,h) \in C_i \times C_j | gh \in C_k\}$$

for any $i, j, k \in \{0, 1, \ldots, d\}$ with $S_{ijk} \neq \emptyset$, we say that G is triply transitive. Note that, since $G \times G = \cup_{i,j,k} S_{ijk}$ and $\dim T_0 = |\{(i,j,k) | S_{ijk} \neq \emptyset\}|$, G is triply transitive if and only if $\dim T_0 = \dim \tilde{T}$. In this case $T_0 = T = \tilde{T}$ holds. We call the finite group G triply regular if $T_0 = T$, and dually triply regular if $T_0^* = T$. Since T_0 or T_0^* generates T as an algebra, G is triply regular (resp. dually triply regular) precisely when the subspace T_0 (resp. T_0^*) is a subalgebra.

A combinatorial meaning of the triple regularity is as follows. Given i,j,k,l,m,n, the size of the set

$$\{z \in C_n | (y,z) \in R_l, (x,z) \in R_m\}$$

depends only on i,j,k,l,m,n, and is independent of the choice of $(x,y) \in S_{ijk}$. This property, when reformulated for association schemes, plays a central role in the theory of spin models (see [4], [5]).

If G is abelian, then $\dim T_0 = |G|^2$, so that G is triply transitive.

Examples. (i) Let $G = A_4$ be the alternating group on four letters. G is triply regular but not triply transitive.

(ii) All finite groups of order 16 are triply transitive.

As for the dual triple regularity, we give a sufficient condition in terms of character products.

Theorem 3 *Let $\chi_0, \dots, \chi_d$ be the complex irreducible characters of the finite group G. If $\chi_i\chi_j$ is multiplicity-free for any i,j, then $T_0^* = \tilde{T}$ holds, in particular, G is dually triply regular. Conversely, $T_0^* = \tilde{T}$ implies that $\chi_i\chi_j$ is multiplicity-free for any i,j.*

Proof. Write $\chi_i\chi_j = \sum_{k=0}^d N_{ij}^k \chi_k$. Then we have

$$\begin{aligned}
\sum_{i,j,k} (N_{ij}^k)^2 &= \sum_{i,j=0}^d (\chi_i\chi_j, \chi_i\chi_j) \\
&= \frac{1}{|G|} \sum_{g\in G} \Big(\sum_{i=0}^d \chi_i(g)\overline{\chi_i(g)}\Big)\Big(\sum_{j=0}^d \chi_j(g)\overline{\chi_j(g)}\Big) \\
&= \frac{1}{|G|} \sum_{g\in G} |C_G(g)|^2 \\
&= \sum_{i=0}^d \frac{|G|}{|C_i|}
\end{aligned}$$

Thus, if $N_{ij}^k \in \{0,1\}$, then

$$\begin{aligned}
\dim \tilde{T} &= \sum_{i=0}^d |G|/|C_i| \\
&= \sum_{i,j,k} N_{ij}^k \\
&= |\{(i,j,k) | N_{ij}^k \neq 0\}| \\
&= |\{(i,j,k) | q_{ij}^k \neq 0\}| \\
&= \dim T_0^*
\end{aligned}$$

as desired. The converse can also be seen from the above equalities. □

Proposition 4 *Let G_1 and G_2 be finite groups.*

(i) *If G_1 and G_2 are triply transitive, so is $G_1 \times G_2$.*

(ii) *If G_1 and G_2 are triply regular, so is $G_1 \times G_2$.*

(iii) *If G_1 and G_2 are dually triply regular, so is $G_1 \times G_2$.*

Proof. (i) This follows immediately from the definition.

(ii) If G_1 and G_2 are triply regular, then $T_0(G_1)$ and $T_0(G_2)$ are subalgebras. Since $T_0(G_1 \times G_2) = T_0(G_1) \otimes T_0(G_2)$, it follows that $G_1 \times G_2$ is triply regular.

(iii) Similar as (ii). $\square$

Theorem 5 *Let $D_{2n} = \langle \sigma, \tau | \sigma^n = 1,\ \tau^2 = 1,\ \tau\sigma\tau = \sigma^{-1} \rangle$ be the dihedral group of order $2n$. Then D_{2n} is triply transitive and dually triply regular.*

Proof. First suppose that n is odd, say $n = 2m+1$. Then the conjugacy classes of D_{2n} are $C_0 = \{e\}$, $C_i = \{\sigma^i, \sigma^{-i}\}$ $(1 \le i \le m)$ and $C_{m+1} = \tau\langle\sigma\rangle$. Thus $\dim \tilde{T} = 2m^2 + 5m + 4$. In order to compute $\dim T_0$, let us consider the product $C_i C_j$. If $1 \le i \le m$, then

$$C_i C_j = \begin{cases} C_i & \text{if } j = 0 \\ C_0 \cup \{\sigma^{2i}, \sigma^{-2i}\} & \text{if } j = i \\ \{\sigma^{i+j}, \sigma^{-i-j}\} \cup \{\sigma^{i-j}, \sigma^{j-i}\} & \text{if } 1 \le j \le m \text{ and } j \ne i \\ C_{m+1} & \text{if } j = m+1. \end{cases}$$

Also

$$C_{m+1} C_j = \begin{cases} C_{m+1} & \text{if } 0 \le j \le m \\ C_0 \cup \cdots \cup C_m & \text{if } j = m+1. \end{cases}$$

We can easily see that the number of triples (i,j,k) with $(C_i C_j) \cap C_k \ne \emptyset$ is $(m+2) + m(2m+2) + (2m+2) = 2m^2 + 5m + 4 = \dim \tilde{T}$, so that D_{2n} is triply transitive.

To show that D_{2n} is dually triply regular, it suffices to prove that $\chi_i \chi_j$ is multiplicity-free, where $\{\chi_0, \ldots, \chi_{m+1}\}$ is the set of complex irreducible characters of D_{2n}. We may assume $\chi_i(1) = \chi_j(1) = 2$, otherwise one of χ_i or χ_j has degree 1, so that $\chi_i \chi_j$ is irreducible. But all irreducible characters of degree 2 are obtained by inducing a linear character of the subgroup $\langle\sigma\rangle$ to D_{2n}. It is now straightforward to check that $\chi_i \chi_j$ is multiplicity-free. Hence D_{2n} is dually triply regular by Theorem 3.

Next suppose that n is even, say $n = 2m$. Then the conjugacy classes of D_{2n} are $C_0 = \{e\}$, $C_i = \{\sigma^i, \sigma^{-i}\}$ $(1 \le i \le m-1)$, $C_m = \{\sigma^m\}$, $C_{m+1} = \tau\langle\sigma^2\rangle$, and $C_{m+2} = \tau\sigma\langle\sigma^2\rangle$. Thus $\dim \tilde{T} = 2m^2 + 6m + 8$. A tedious calculation similar to the case $n = 2m+1$ establishes $\dim T_0 = \dim T_0^* = 2m^2 + 6m + 8$, hence D_{2n} is triply transitive and dually triply regular. $\square$

We have computed $\dim T_0$, $\dim T_0^*$, and $\dim \tilde{T}$ for all nonabelian indecomposable finite groups of order at most 100 using GAP [6]. The results are tabulated in the Appendix. We have not found any group for which $\dim T_0 = \dim \tilde{T} > \dim T_0^*$ holds.

To conclude this section, we give a list of indecomposable finite groups of order at most 24 which are not triply transitive. Balmaceda and Oura [1] has determined the Terwilliger algebra for $G = S_5$ and A_5.

	$\dim T_0$	$\dim T_0^*$	$\dim T$	$\dim \tilde{T}$
A_4	19	19	19	22
5.4	29	29	29	37
7.3	35	35	37	41
$SL(2,3)$	75	73	75	76
S_4	42	43	43	43

3 Relationship with the quantum double

Let $\tilde{\mathcal{A}}$ be the complex vector space with basis $G \times G \times G$, and define the multiplication in $\tilde{\mathcal{A}}$ by

$$(x, g, a)(y, h, b) = \delta_{x^{-1}ga,h}(xy, g, ab)$$

and extend it linearly to $\tilde{\mathcal{A}}$. Then $\tilde{\mathcal{A}}$ becomes an associative algebra. The subalgebra of $\tilde{\mathcal{A}}$ defined by

$$\mathcal{D} = \langle (h, g, h) | g, h \in G \rangle_{\mathbf{C}} \subset \tilde{\mathcal{A}}.$$

is known as the quantum double of the finite group G (precisely speaking, the quantum double is defined for a Hopf algebra, and $\mathcal{D}$ is the quantum double of the group Hopf algebra of G, see [3]). On the other hand, let

$$\mathcal{T} = \langle (1, g, h) | g, h \in G \rangle_{\mathbf{C}} \subset \tilde{\mathcal{A}},$$

and denote by $\mathcal{T}^G$ the G-fixed subspace of $\mathcal{T}$. Then $\mathcal{T}^G$ is isomorphic to the centralizer algebra $\tilde{T}$ defined in Section 2. Therefore, $\tilde{\mathcal{A}}$ is an algebra containing both the quantum double $\mathcal{D}$ and the Terwilliger algebra T.

References

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