Distance-regular graphs related to the binary Golay code and their spherical representation

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A distance-regular graph (DRG) is a connected finite undirected graph Γ of diameter d such that

is well-defined. More precisely,

$$
\begin{aligned}
a_i &= |\{y \mid d_\Gamma(x,y) = 1,\ d_\Gamma(y,z) = i\}|, \\
b_i &= |\{y \mid d_\Gamma(x,y) = 1,\ d_\Gamma(y,z) = i+1\}|, \\
c_i &= |\{y \mid d_\Gamma(x,y) = 1,\ d_\Gamma(y,z) = i-1\}|,
\end{aligned}
$$

are independent of x, z as long as $d_\Gamma(x,z) = i$.

The numbers a_i, b_i, c_i are called the parameters or the intersection numbers of the DRG Γ.

Soicher (1995) discovered a DRG with 672 vertices:

Meixner (1991) discovered a DRG with 1344 vertices:

$$A_1 = \text{the adjacency matrix of } \Gamma$$

$$(A_1)_{xy} = \begin{cases} 1 & x \text{ is adjacent to } y \\ 0 & \text{otherwise} \end{cases}$$

Let λ be an eigenvalue of A_1 (also called an eigenvalue of Γ), $n = |V(\Gamma)|$.

$\mathbb{R}^n$: the vector space with unit vectors e_x indexed by $V(\Gamma)$.

$W_\lambda = \{v \in \mathbb{R}^n \mid A_1 v = \lambda v\}$.

$\pi_\lambda : \mathbb{R}^n \to W_\lambda$: orthogonal projection.

The spherical representation of Γ is

$$\{\pi_\lambda(e_x) \mid x \in V(\Gamma)\}$$

The binary Golay code is the subspace of $\mathbb{F}_2^{23}$ spanned by the row vectors of the following matrix.

$$\begin{pmatrix}
10000000000011111111111\\
01000000000010100011101\\
00100000000011010001110\\
00010000000001101000111\\
00001000000010110100011\\
00000100000011011010001\\
00000010000011101101000\\
00000001000001110110100\\
00000000100000111011010\\
00000000010000011101101\\
00000000001010001110110\\
00000000000101000111011
\end{pmatrix}$$

The truncated binary Golay code G_{22} is the subspace of $\mathbb{F}_2^{22}$ spanned by the row vectors of the matrix obtained from the above matrix by deleting one column.

1344 vectors of weight 11 in the truncated binary Golay code G_{22} consists of 672 complementary pairs.

Meixner's graph $\tilde{\Gamma}$ is defined by:

$V(\tilde{\Gamma}) =$ 1344 vectors of weight 11 in G_{22}.

$V(\tilde{\Gamma}) \ni u, v$ are adjacent if and only if
(i) $\mathrm{wt}(u * v) = 3$, or
(ii) $\mathrm{wt}(u * v) = 7$ and $\nexists$ hexad h with $\mathrm{wt}(h * u * v) = 5$, or
(iii) $\mathrm{wt}(u * v) = 6$ and $\exists$ hexad h with $\mathrm{wt}(h * u * v) = 5$.

where $*$ denotes the entrywise multiplication.

$|\mathrm{Aut}(G_{22}) : M_{22}| = 2$.

M_{22} has two orbits on the set of 1344 vectors of weight 11. Soicher's graph is the induced subgraph of $\tilde{\Gamma}$ on either one of the orbits of M_{22}.

Spherical representation.

$\tilde{\Gamma}$ has eigenvalue 44 with multiplicity 56.

$$96(\pi_{44}(e_x), \pi_{44}(e_y)) = 4, 1, 0, -1, -4$$

according as $d_{\tilde{\Gamma}}(x, y) = 0, 1, 2, 3, 4$.

Γ has eigenvalue 26 with multiplicity 55.

$$672(\pi_{26}(e_x), \pi_{26}(e_y)) = 55, 13, -1, -15$$

according as $d_{\tilde{\Gamma}}(x, y) = 0, 1, 2, 3$.

Also, Γ has the Frobenius eigenvalue 110 with multiplicity 1.

$672(\pi_{110}(e_x), \pi_{110}(e_y)) = 1$ for all $x, y \in \Gamma$.

Define $\pi : V(\Gamma) \to \mathbb{R}^{56} = W_{110} \oplus W_{26}$ by

$$\pi = \pi_{110} \oplus \pi_{26}.$$

Then

$$672(\pi(e_x), \pi(e_y)) = 4, 1, 0, -1,$$

according as $d_\Gamma(x, y) = 0, 1, 2, 3.$

Note

$$(\pi(e_x), -\pi(e_y)) = -4, -1, 0, 1,$$

according as $d_\Gamma(x, y) = 0, 1, 2, 3.$

It turns out that

$$\{\pm\pi(e_x) \mid x \in V(\Gamma)\}$$

gives the spherical representation π_{44} of $\tilde{\Gamma}$ (up to scaling).

Combinatorially

$$\tilde{\Gamma} =$$

In terms of the adjacency matrix:

$$\tilde{A}_1 = \begin{pmatrix} A_1 & A_3 \\ A_3 & A_1 \end{pmatrix}$$

where

$$(A_3)_{xy} = \begin{cases} 1 & \text{if } d_\Gamma(x,y) = 3, \\ 0 & \text{otherwise} \end{cases}$$

Theorem. $\tilde{A}_1$ is the adjacency matrix of Meixner's graph $\tilde{\Gamma}$.

Theorem. Let Γ be a non-bipartite DRG of diameter 3, Let A_i $(i = 1, 3)$ be the matrix defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } d_\Gamma(x,y) = i, \\ 0 & \text{otherwise} \end{cases}$$

Then the matrix

$$\tilde{A}_1 = \begin{pmatrix} A_1 & A_3 \\ A_3 & A_1 \end{pmatrix}$$

is the adjacency matrix of a DRG if and only if Γ has parameters

$$\begin{aligned} b_0 &= (pq + p + q)(q + 1)/2 \\ b_1 &= (p + 1)(q + 2)(q - 1)/2 \\ b_2 &= q(p + q)/4 \\ c_2 &= (p + q)(q + 2)/4 \\ c_3 &= q(p + 1)(q + 1)/2 \end{aligned}$$

for some integers p, q.

The parameters of the resulting graph $\tilde{\Gamma}$ with adjacency matrix $\tilde{A}_1$ coincides with a family of dual bipartite Q-polynomial DRGs given by Dickie and Terwilliger (1996).

p	q	v	comments
2	2	35	$J(7,3)$
4	2	64	Halved 7-cube
8	4	672	Soicher
9	3	378	$O(7,3)$?
16	4	1408	
20	4	1782	Suz?