An Introduction to Designs in Spheres and Complex Projective Spaces

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Spherical Designs

Why can't we place 5 points on a sphere in a nice way, even though we can easily do the same for 4 points (tetrahedron) or for 6 points (octahedron)?

We will answer this question rigorously by defining spherical design. There is no spherical 2-design in $\mathbb{R}^3$ of 4 points, or of 6 points, but not of 5 points.

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Definition of Spherical Design

Definition

Let d be a positive integer. Let $\Omega_d = \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\| = 1\}$ be the unit sphere in $\mathbb{R}^d$.

A *spherical t-design* is a finite nonempty subset X of Ω_d satisfying

$$\frac{1}{\mathrm{volume}(\Omega_d)} \int_{\Omega_d} f(\xi) d\xi = \frac{1}{|X|} \sum_{\mathbf{x} \in X} f(\mathbf{x}) \tag{1}$$

for all polynomial functions f of degree at most t.

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Existence of Spherical 2-Designs

Theorem (Mimura, 1990)

Let n, d be positive integers with $d \geq 2$. Then there exists a spherical 2-design of n points in $\mathbb{R}^d$ unless $n \leq d$ or $n = d + 2$ is odd.

In particular, there is no spherical 2-design of 5 points in $\mathbb{R}^3$.
If n or d is even, then the construction is easy.
If both n and d are odd, we will give a construction which is much simpler than Mimura's.

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Angle Set of Spherical Design

The angle set of a finite set $X \subset \Omega_d$ is

$$A(X) = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in X,\ \mathbf{x} \neq \mathbf{y}\} \subset [-1, 1).$$

If we regard it as a multiset, then the property of being a spherical t-design can be described in terms of the angle set.

Theorem (Delsarte-Goethals-Seidel)

A finite set $X \subset \Omega_d$ is a spherical t-design if and only if

$$\sum_{\mathbf{x}, \mathbf{y} \in X} P_k((\mathbf{x}, \mathbf{y})) = 0 \quad \text{for } k = 1, 2, \ldots, t,$$

where $P_k(x)$ $(k = 1, 2, \ldots)$ are Gegenbauer polynomials.

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Designs in Complex Projective Spaces

Let $\Omega_d(\mathbb{C})$ denote the set of vectors of $\mathbb{C}^d$ of unit length. The complex projective space P^{d-1} is the quotient set of $\Omega_d(\mathbb{C})$, by the equivalence relation

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = e^{\sqrt{-1}\theta}\mathbf{y} \quad \text{for some } \theta \in \mathbb{R}.$$

Definition

A *t-design in* P^{d-1} is a finite nonempty subset X of P^{d-1} satisfying

$$\int_{P^{d-1}} f(\xi)d\xi = \frac{1}{|X|}\sum_{x\in X} f(x) \tag{2}$$

for all $f \in \bigoplus_{k=0}^{t} \mathrm{Hom}(k)$, where $d\xi$ denotes the unique normalized Haar measure invariant under the unitary group $U(d,\mathbb{C})$, and $\mathrm{Hom}(k)$ will be defined later.

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Examples of 2-designs

- $d+1$ mutually unbiased bases in $\mathbb{C}^d$
- Symmetric informationally complete positive operator-valued measure (SIC-POVM).

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