An Introduction to Designs in Spheres and Complex Projective Spaces

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2006年9月14日

1 Characterization of spherical 2-designs

Definition 1. Let d be a positive integer. Let $\Omega_d = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^d$. A *spherical t-design* is a finite nonempty subset (multiset) X of Ω_d satisfying

$$\frac{1}{\text{volume}(\Omega_d)} \int_{\Omega_d} f(\xi) d\xi = \frac{1}{|X|} \sum_{x \in X} f(x) \tag{1}$$

for all polynomial functions f of degree at most t.

Let $\text{Hom}(k)$ denote the linear space of homogeneous polynomials of degree k.

$$\text{Hom}(1) = \langle x_i \mid 1 \le i \le d \rangle,$$
$$\text{Hom}(2) = \langle x_i x_j, x_k^2 \mid 1 \le i < j \le d,\ 1 \le k \le d \rangle.$$

For a $d \times n$ matrix, put $W = \sqrt{d/n} X = (w_{ik})$.

Lemma 2. The column vectors of X form a spherical 2-design in $\mathbb{R}^d$ if and only if

$$\sum_{i=1}^{d} w_{ik}^2 = \frac{d}{n} \quad (1 \le k \le n), \tag{C0}$$

$$\sum_{k=1}^{n} w_{ik} = 0 \quad (1 \le i \le d), \tag{C1}$$

$$\sum_{k=1}^{n} w_{ik} w_{jk} = 0 \quad (1 \le i < j \le d), \tag{C2}$$

$$\sum_{k=1}^{n} w_{ik}^2 = 1 \quad (1 \le i \le d). \tag{C3}$$

Let

$$v_0 = \sqrt{\frac{1}{n}}(1, \dots, 1) \in \mathbb{R}^n. \tag{2}$$

Lemma 3. Assume that the matrix W satisfies (C0). Then W satisfies (C1)–(C3) if and only if there exists an $(n-d-1) \times n$ matrix $W' = (w'_{ik})$ such that

$$U = \begin{pmatrix} v_0 \\ W \\ W' \end{pmatrix} \tag{3}$$

is an orthogonal matrix of size n. In particular, $n \geq d+1$.

Lemma 4. The existence of a spherical 2-design of n points in $\mathbb{R}^d$ implies the existence of a spherical 2-design of n points in $\mathbb{R}^{n-d-1}$. In particular, if d is odd, then there is no spherical 2-design of $d+2$ points in $\mathbb{R}^d$.

Lemma 5. Suppose that $X = \begin{pmatrix} \boldsymbol{x}_1, \dots, \boldsymbol{x}_n \end{pmatrix}$ is a $d \times n$ matrix with entries in $\mathbb{R}$, and $\|\boldsymbol{x}_i\| = 1$ for $1 \leq i \leq n$. Then the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ form a spherical 2-design in $\mathbb{R}^d$ if and only if

$$\sum_{i,j=1}^{n} (\boldsymbol{x}_i, \boldsymbol{x}_j) = 0, \tag{4}$$

$$\sum_{i,j=1}^{n} (\boldsymbol{x}_i, \boldsymbol{x}_j)^2 = \frac{n^2}{d}. \tag{5}$$

Lemma 5 implies that the property of being a spherical 2-design is completely described in terms of its "angle set":

$$A(X) = \{(\boldsymbol{x}_i, \boldsymbol{x}_j) \mid 1 \leq i, j \leq n\},$$

if we regard it as a multiset. This is true in general, for spherical t-designs.

Definition 6. The Gegenbauer polynomials $\{P_k\}_{k=0}^{\infty}$ are defined by

$$P_0(x) = 1, \qquad P_1(x) = dx,$$
$$\frac{k+1}{d+2k} P_{k+1}(x) = xP_k(x) - \frac{d+k-3}{d+2k-4} P_{k-1}(x) \quad (k = 1, 2, \dots).$$

For example,

$$P_2(x) = \frac{d+2}{2}(dx^2 - 1).$$

Thus, (5) is equivalent to

$$\sum_{i,j=1}^{n} P_2((\boldsymbol{x}_i, \boldsymbol{x}_j)) = 0,$$

while obviously, (4) is equivalent to

$$\sum_{i,j=1}^{n} P_1((\boldsymbol{x}_i, \boldsymbol{x}_j)) = 0.$$

Theorem 7 (Delsarte-Goethals-Seidel)**.** A finite set $X \subset \Omega_d$ is a spherical t-design if and only if

$$\sum_{\boldsymbol{x},\boldsymbol{y}\in X} P_k((\boldsymbol{x},\boldsymbol{y})) = 0 \quad \text{for } k = 1, 2, \dots, t.$$

2 Construction

Lemma 8. Let n be a positive integer, and let $\zeta = \exp(2\pi\sqrt{-1}/n)$. Define $u_k \in \mathbb{C}^n$ ($k \in \mathbb{Z}$) by

$$u_k = (\zeta^k, \zeta^{2k}, \dots, \zeta^{nk}),$$

and define $c_k, s_k \in \mathbb{R}^n$ ($k \in \mathbb{Z}$, $0 \le k \le n/2$) by

$$c_k + \sqrt{-1} s_k = r_k u_k, \tag{6}$$

where $r_0 = \sqrt{1/n}$, $r_{n/2} = \sqrt{1/n}$ if n is even, $r_k = \sqrt{2/n}$ for $1 \le k < n/2$. Then the set

$$\{c_k \mid 0 \le k \le n/2\} \cup \{s_k \mid 1 \le k < n/2\} \tag{7}$$

forms an orthonormal basis of $\mathbb{R}^n$.

Lemma 9. Let m, n be positive integers such that $1 \le m < n/2$. Let W be the $2m \times n$ matrix defined by

$$W = \begin{pmatrix} c_1 \\ \vdots \\ c_m \\ s_1 \\ \vdots \\ s_m \end{pmatrix}.$$

When n is even, let W' be the $(2m+1) \times n$ matrix defined by

$$W' = \begin{pmatrix} c_1 \\ \vdots \\ c_m \\ s_1 \\ \vdots \\ s_m \\ c_{n/2} \end{pmatrix}.$$

Then each of W and W' satisfies the conditions (C0)–(C3).

In particular, if nd is even and $n \ge d + 1$, then there exists a spherical 2-design of n points in $\mathbb{R}^d$.

Lemma 10. Let n, d be odd positive integers satisfying

$$n \ge 2d + 1 \ge 7. \tag{8}$$

Then there exists a spherical 2-design of n points in $\mathbb{R}^d$.

Theorem 11 (Mimura)**.** Let n, d be positive integers with $d \geq 2$. The following are equivalent.

(i) there exists a spherical 2-design of n points in $\mathbb{R}^d$,

(ii) $n \geq d+1$, and $n \neq d+2$ if d is odd.

3 Designs in complex projective spaces

Let $\Omega_d(\mathbb{C})$ denote the set of vectors of $\mathbb{C}^d$ of unit length. The complex projective space P^{d-1} is the quotient set of $\Omega_d(\mathbb{C})$, by the equivalence relation

$$\boldsymbol{x} \sim \boldsymbol{y} \iff \boldsymbol{x} = e^{\sqrt{-1}\theta}\boldsymbol{y} \quad \text{for some } \theta \in \mathbb{R}.$$

We denote the equivalence class containing $\boldsymbol{x} \in \Omega_d(\mathbb{C})$ by $[\boldsymbol{x}]$. Alternatively, one can regard $[\boldsymbol{x}]$ as a Hermitian matrix

$$|\boldsymbol{x}\rangle\langle\boldsymbol{x}| = (x_i\overline{x_j}) \in M_d(\mathbb{C}).$$

For every $d \times d$ Hermitian idempotent matrix A of rank 1, there exists $\boldsymbol{x} \in \Omega_d(\mathbb{C})$ such that $A = |\boldsymbol{x}\rangle\langle\boldsymbol{x}|$. Therefore there is a bijection

$$\begin{aligned} P^{d-1} &\to \{A \in M_d(\mathbb{C}) \mid A = \bar{A}^T = A^2, \ \operatorname{rank} A = 1\} \\ [\boldsymbol{x}] &\mapsto |\boldsymbol{x}\rangle\langle\boldsymbol{x}|. \end{aligned}$$

Under the above identification, however, one can consider polynomial functions in terms of coordinates of the matrix $|\boldsymbol{x}\rangle\langle\boldsymbol{x}|$. These are polynomials homogeneous of degree k in the variables $x_1, \dots, x_d$, and homogeneous of degree k in the variables $\overline{x_1}, \dots, \overline{x_d}$. So we define $\operatorname{Hom}(k)$ to be the linear space of such functions.

Definition 12. A *t-design in* P^{d-1} is a finite nonempty subset X of P^{d-1} satisfying

$$\int_{P^{d-1}} f(\xi) d\xi = \frac{1}{|X|} \sum_{x \in X} f(x) \tag{9}$$

for all $f \in \bigoplus_{k=0}^{t} \operatorname{Hom}(k)$, where $d\xi$ denotes the unique normalized Haar measure invariant under the unitary group $U(d, \mathbb{C})$.

For $[\boldsymbol{x}], [\boldsymbol{y}] \in P^{d-1}$, we define their "inner product" to be

$$([\boldsymbol{x}], [\boldsymbol{y}]) = |(\boldsymbol{x}, \boldsymbol{y})|^2 = \operatorname{tr}(|\boldsymbol{x}\rangle\langle\boldsymbol{x}||\boldsymbol{y}\rangle\langle\boldsymbol{y}|).$$

The *angle set* of a finite nonempty subset X of P^{d-1} is

$$A(X) = \{([\boldsymbol{x}, \boldsymbol{y}]) \mid [\boldsymbol{x}] \in X, \ [\boldsymbol{y}] \in X, \ [\boldsymbol{x}] \neq [\boldsymbol{y}]\}.$$

Theorem 13 ([4])**.** For a finite set $X \subset P^{d-1}$, the following are equivalent.

(i) X is a t-design in P^{d-1};

(ii)
$$\frac{1}{|X|^2} \sum_{[\boldsymbol{x}],[\boldsymbol{y}]\in X} ([\boldsymbol{x}],[\boldsymbol{y}])^k = \frac{1}{\binom{d+k-1}{k}} \quad \text{for } 0 \le k \le t. \tag{10}$$

Example 14. Let A, B be two orthonormal bases of $\mathbb{C}^d$. The pair (A, B) is said to be *mutually unbiased* if $|(\boldsymbol{x}, \boldsymbol{y})|^2 = 1/d$ for all $\boldsymbol{x} \in A$ and $\boldsymbol{y} \in B$. Suppose that $A_1, \dots, A_{d+1}$ are orthonormal bases of $\mathbb{C}^d$ which are pairwise mutually unbiased. Let

$$X = \{[\boldsymbol{x}] \in P^{d-1} \mid \boldsymbol{x} \in \bigcup_{i=1}^{d+1} A_i\}.$$

Then X is a 2-design in P^{d-1}.

It is shown in [4, Theorem 4] that, conversely, every 2-design consisting of $d(d+1)$ elements with angle set $\{0, 1/d\}$ in P^{d-1} arises from $d+1$ mutually unbiased bases. Such a 2-design exists whenever d is a prime power [9]. In a different context, Popa [7, Theorem 3.2] already established the existence of such a 2-design whenever d is a prime. Zauner [10] conjectures that such a 2-design does not exist if d is not a prime power.

The following theorem gives an analogue of Fisher's inequality.

Theorem 15. If X is a 2-design in P^{d-1}, then $|X| \ge d^2$. If equality holds, then the angle set of X is $\{1/(d+1)\}$.

A 2-design in P^{d-1} consisting of d^2 points is called a tight 2-design. A tight 2-design is also called a symmetric informationally complete positive operator-valued measure (SIC-POVM), cf [4, 8]. Zauner [10] conjectures that SIC-POVMs exist for all $d \ge 2$. Examples for $d = 3, 8$ are found in [3] and those for $d = 2, 3, 4$ are found in [8].

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