# An extremal problem related to binary singly even self-dual codes

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Given $v, k, \lambda$, find the largest possible size of such a subset $\mathcal{B}$.

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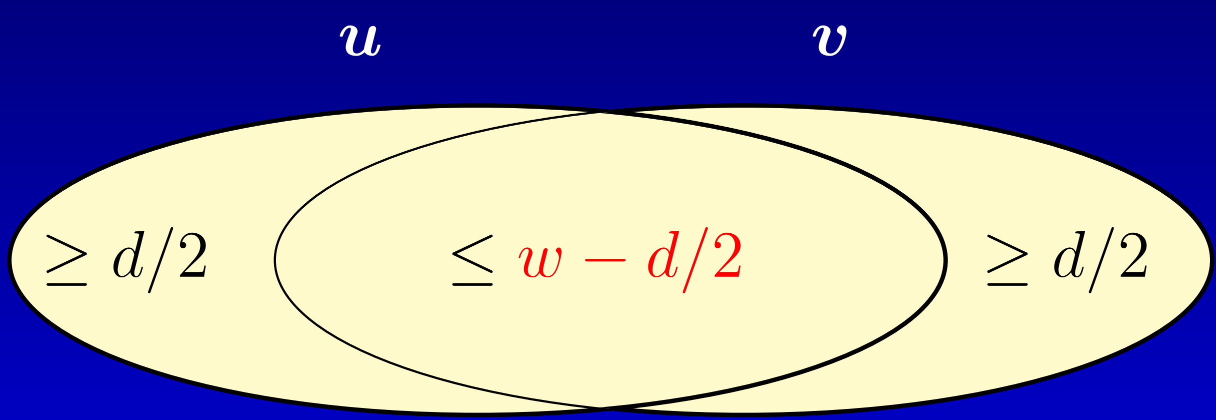

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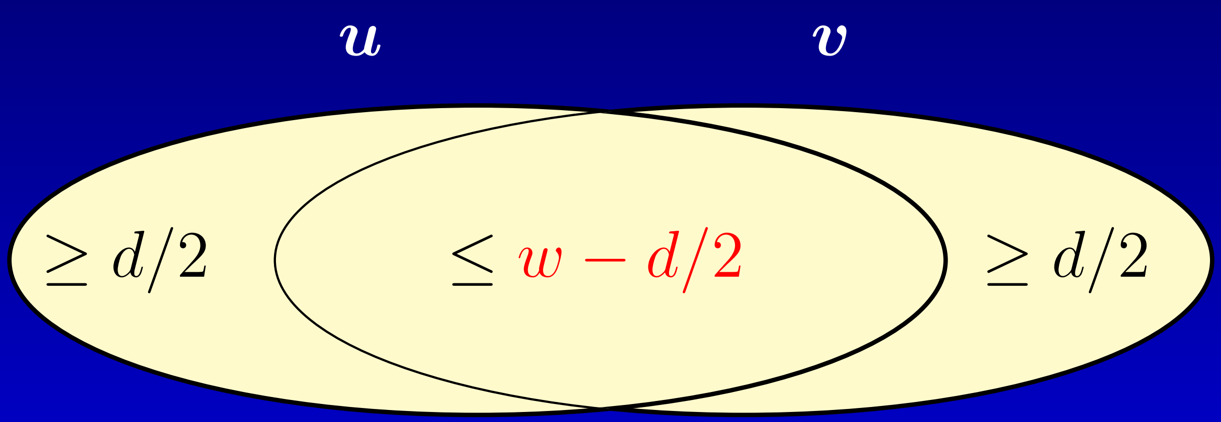


(because $\boldsymbol{u} + \boldsymbol{v} \in C$)

# Linear programming bound

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$$Q_{ij} = \left(\binom{v}{j} - \binom{v}{j-1}\right) \sum_{r=0}^{j} (-1)^r \frac{\binom{i}{r}\binom{j}{r}\binom{v+1-j}{r}}{\binom{k}{r}\binom{v-k}{r}} \quad (0 \leq i, j \leq k).$$

# Example

$v = 62,\ k = 7,\ L = \{0, 1\}$. $|\mathcal{B}|$ is bounded from the above by

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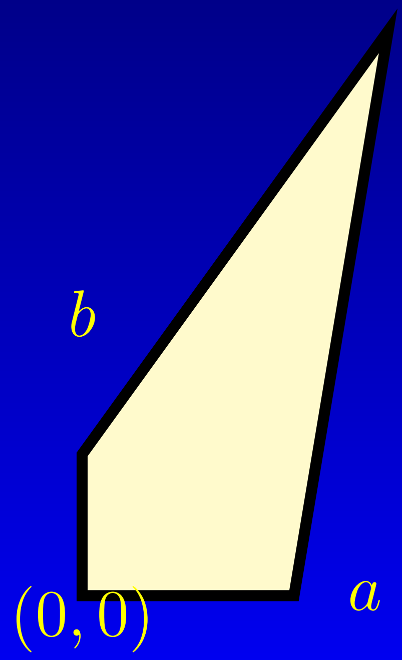

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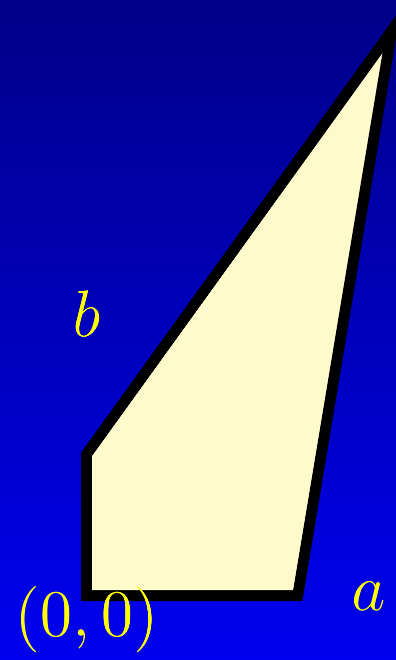

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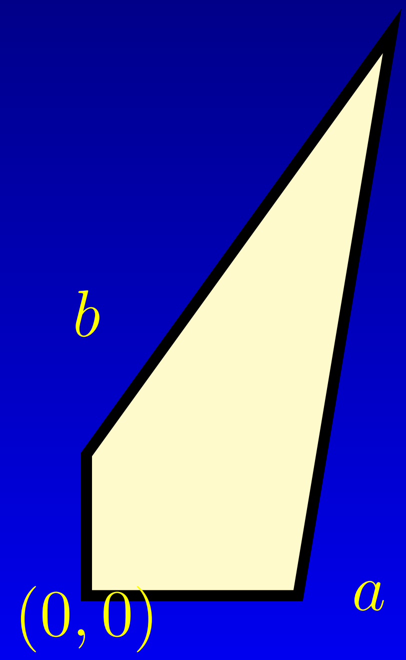

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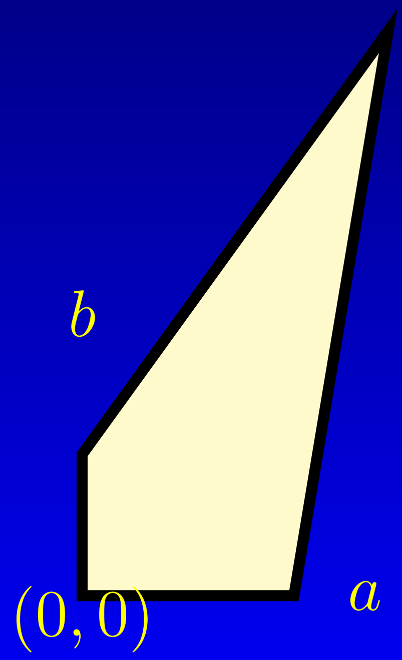


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An $[n, k, d]$ code $C$ is a linear code of length $n$, dimension $k$, and minimum weight $d$.

# Weight enumerator

Let $y$ be an indeterminate. For a binary code $C$ of length $n$, set

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The polynomial $W_C$ is called the weight enumerator of $C$.

# Dual codes

The dual code of a linear code $C$ is

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There are cosets $C_1, C_2, C_3$ of $C_0$ such that
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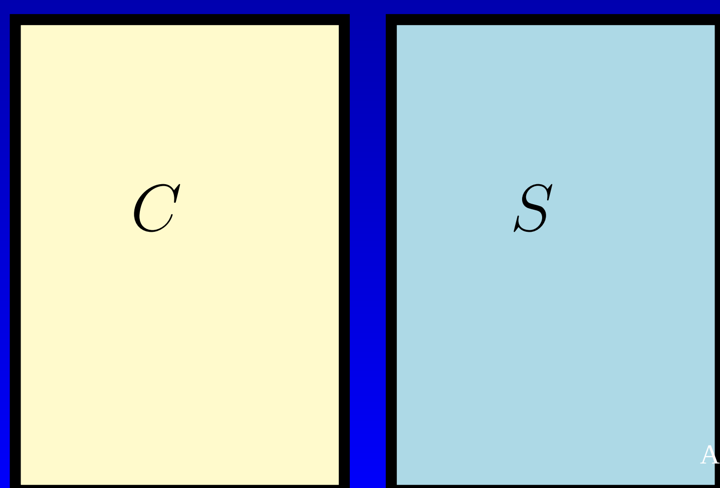

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In particular, $\forall \boldsymbol{u} \in S$,

$$\mathrm{wt}(\boldsymbol{u}) \equiv \frac{n}{2} \pmod 4.$$

# Extremality

The minimum weight $d$ of a self-dual code of length $n$ is bounded from the above by

$$d \leq \begin{cases} 4[n/24]+4 & n \not\equiv 22 \pmod{24}, \\ 4[n/24]+6 & n \equiv 22 \pmod{24}. \end{cases}$$

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Equality imposes strong restrictions on the weight enumerator.

# Self-dual $[62, 31, 12]$ code

$$W_C = 1 + (1860 + 32\beta)y^{12} + (28055 - 160\beta)y^{14} + \cdots ,$$

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where $A(n, d, r)$ is the maximal possible number of binary vectors of length $n$, weight $r$ and Hamming distance at least $d$ apart. This is because $S$ (which is isometric to $C$) has minimum distance $d$.

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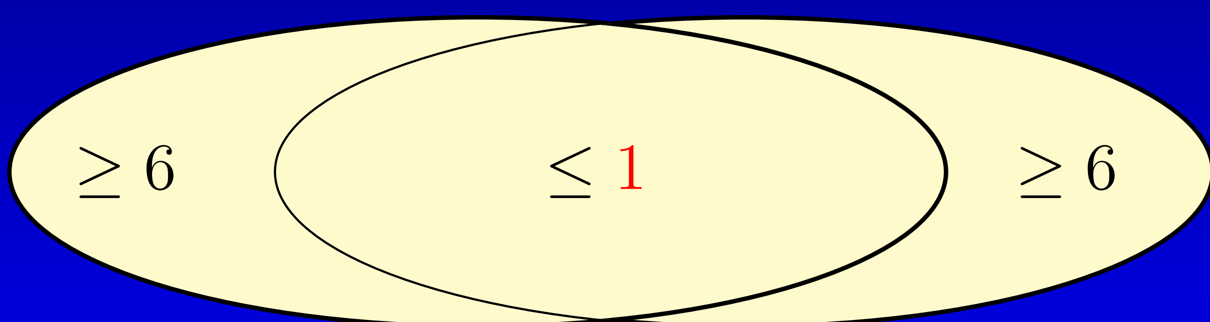

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$$\beta = B_7 \le A(62, 12, 7)$$

Hamming distance at least $12 \iff$ at most $1$-intersecting
We have seen by the linear programming bound that

$$A(62, 12, 7) \le 90,$$

so

$$0 \le \beta \le 90.$$

# Two parts of the shadow

$C$
min. wt. 12

$S$
wt. 7

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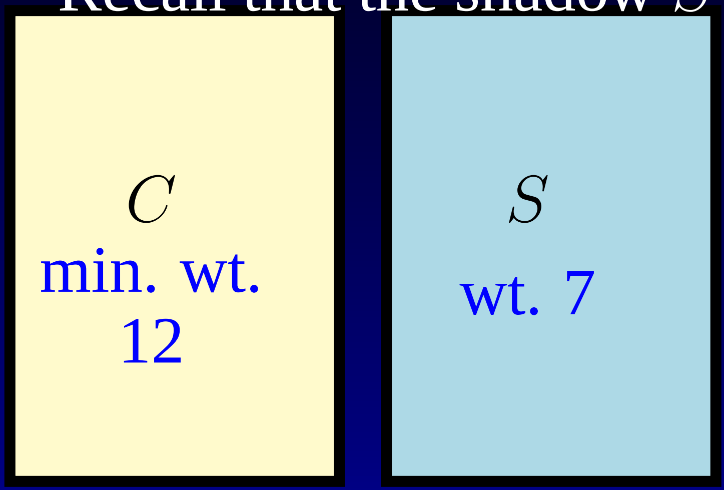
| $C$ | $S$ |
| --- | --- |
| min. wt. 12 | wt. 7 |

$\Longrightarrow$ at most 1-intersecting

# Two parts of the shadow

Recall that the shadow $S$ consists of two cosets $C_1, C_3$ of $C_0$.

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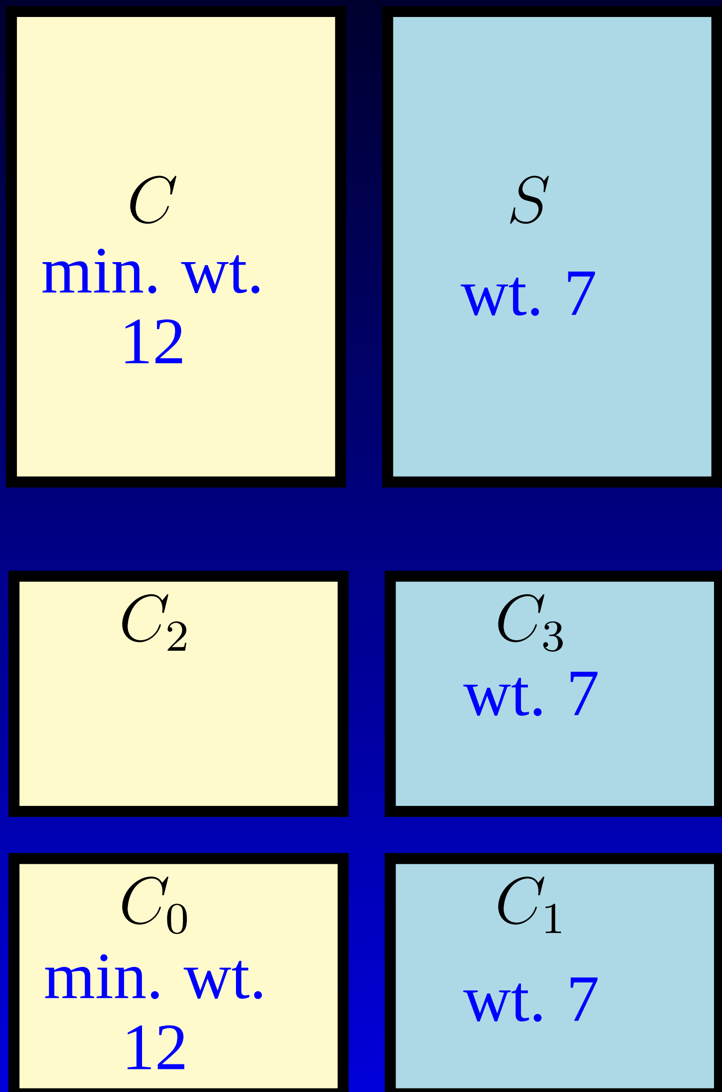


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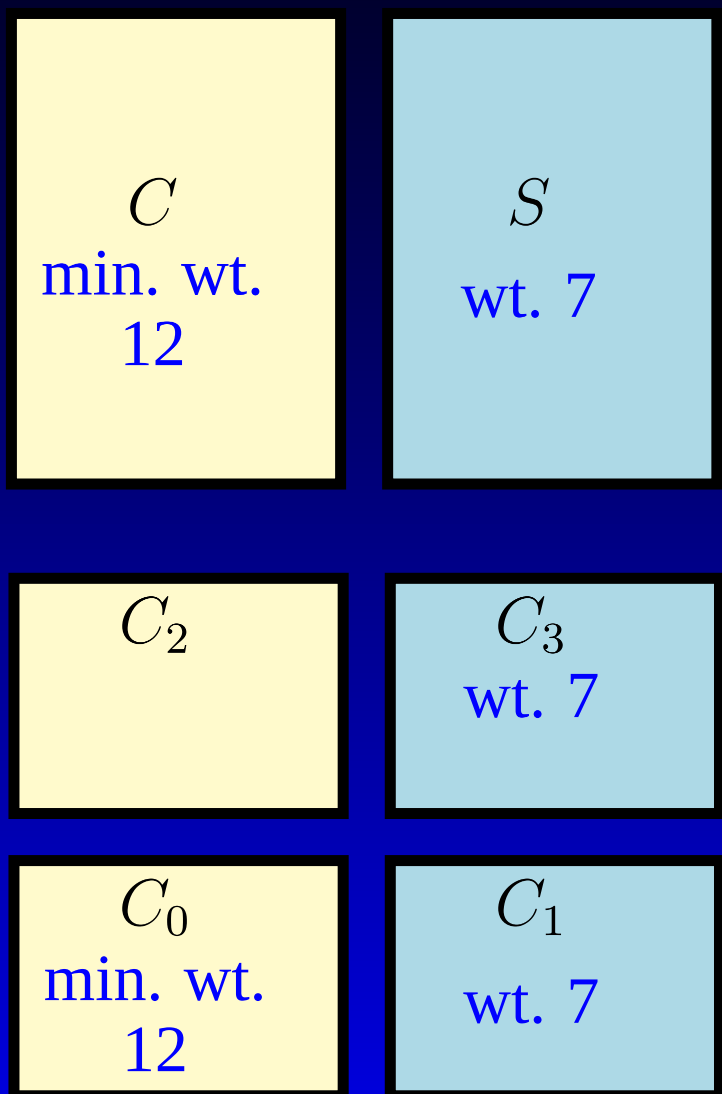


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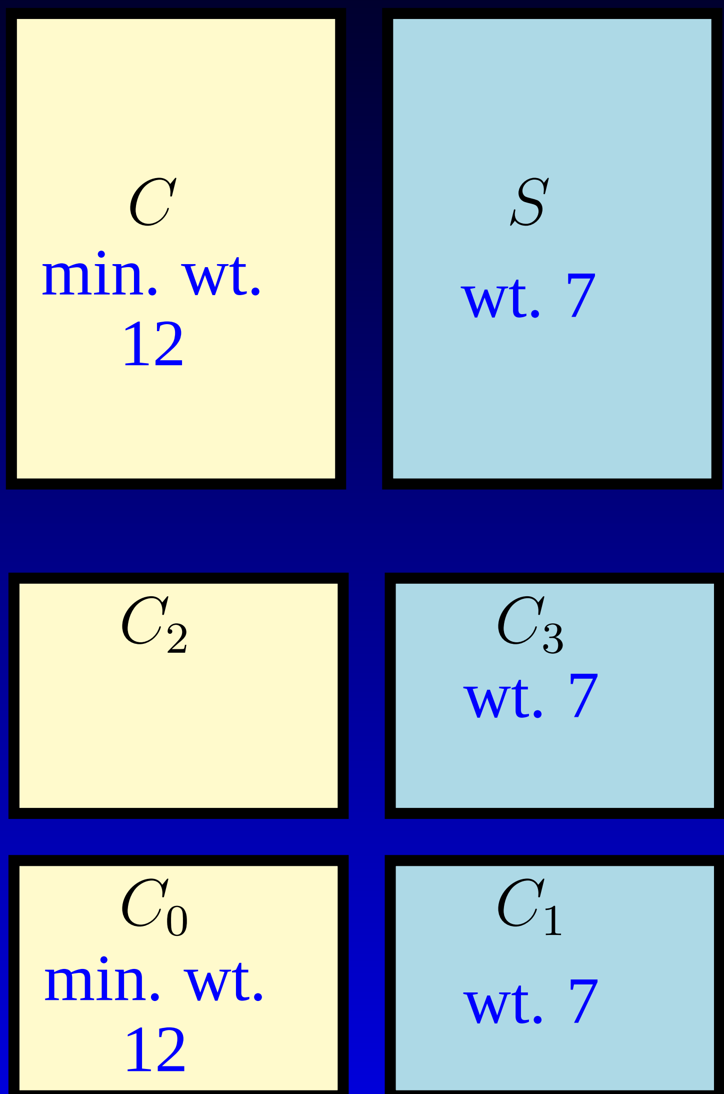


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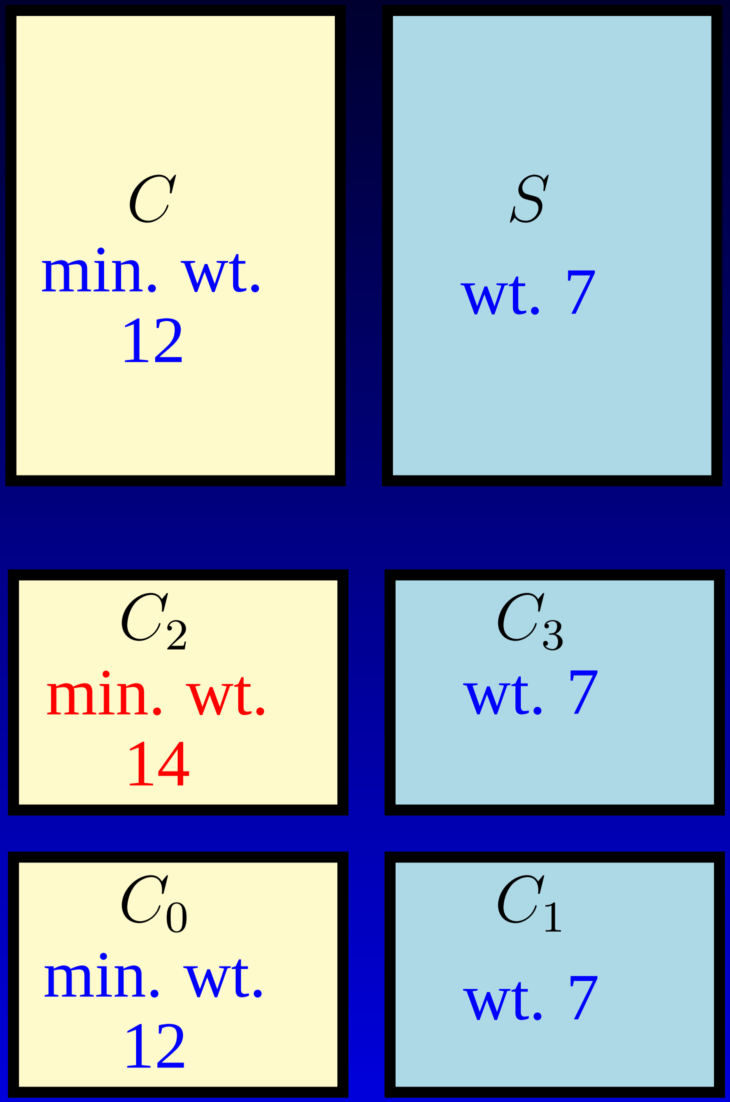


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$C$
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$S$
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Known realizable values of $\beta$: 0,10,15.

(Dontcheva-Harada, 2002)

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Every self-dual $[42, 21, 8]$ code $C$ whose shadow $S$ does not contain a vector of weight $1$ has weight enumerator

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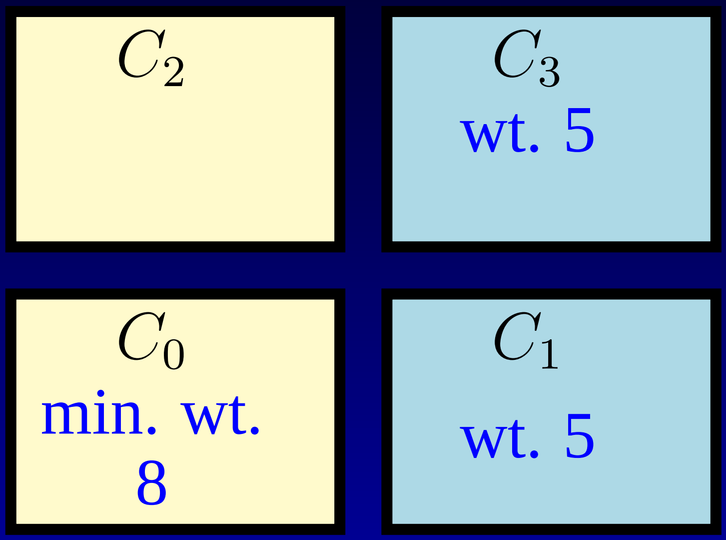

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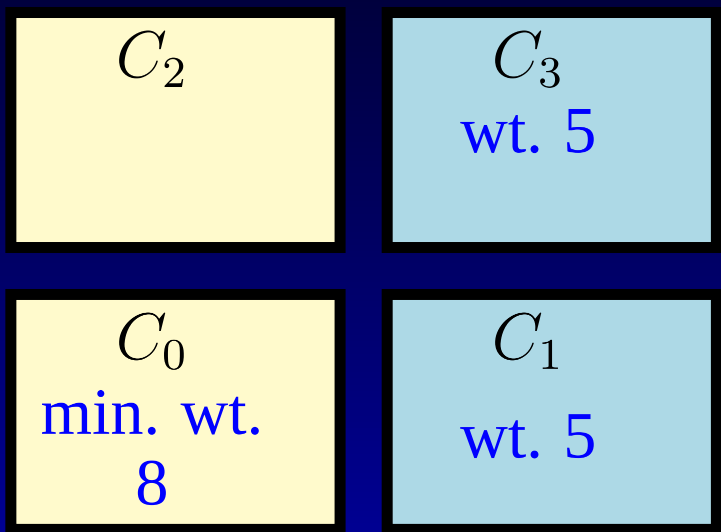


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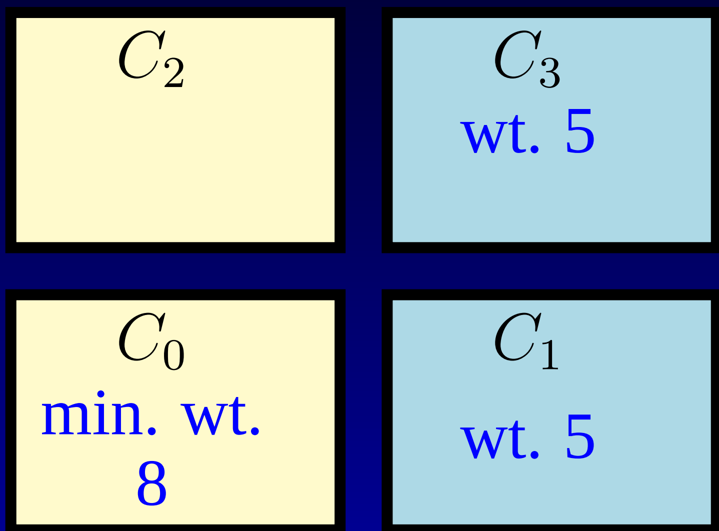


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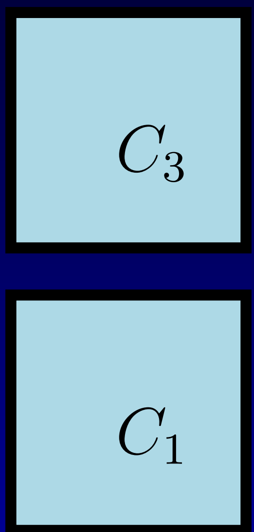

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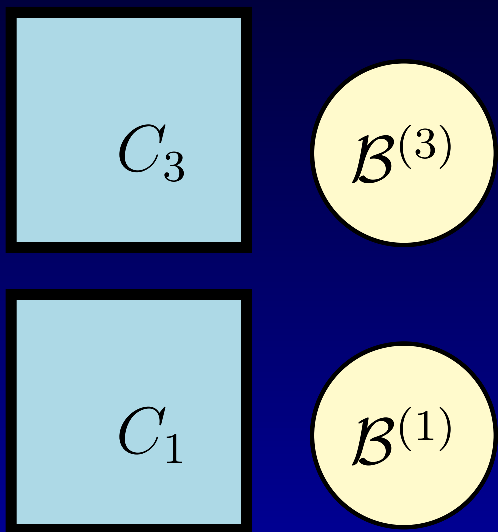


supports of
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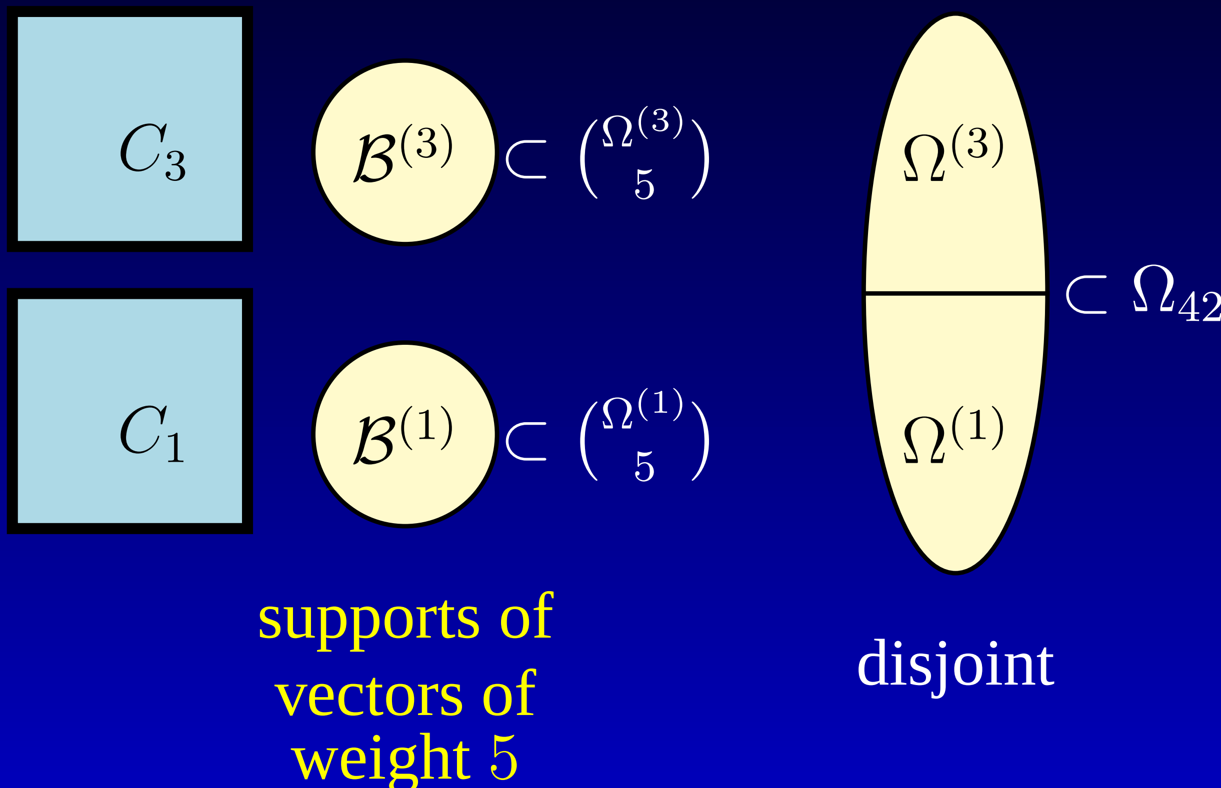

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# Characterization

**Theorem 2.** *There exists a unique binary self-dual* $[42, 21, 8]$ *code with weight enumerator*

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This theorem was obtained recently, and independently, by Stefka Buyuklieva.