Spherical designs and extremal lattices

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Tohoku University
August 15, 2005

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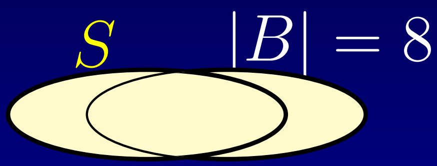

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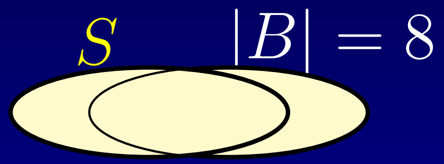

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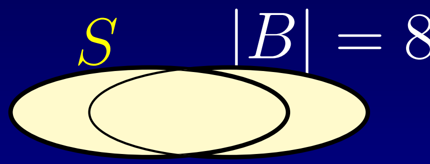

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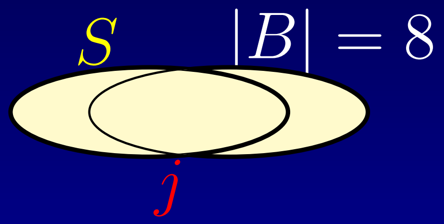

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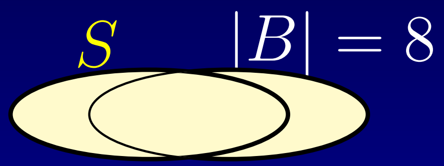

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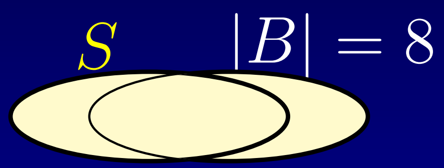

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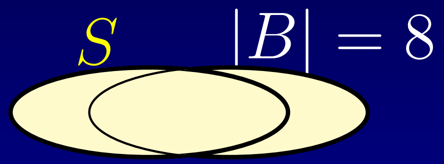

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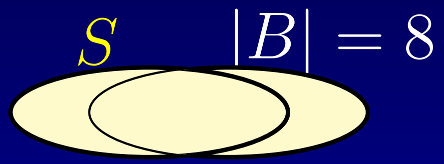

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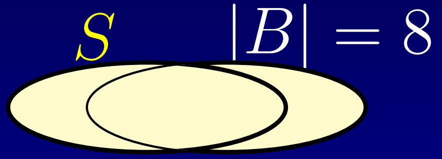

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$$|S|(|S| - 8)(|S| - 12)(|S| - 16)(|S| - 24) = 0.$$

5-$(48, 12, \lambda)$ design

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Assume that every pair of blocks interset at even number of points.

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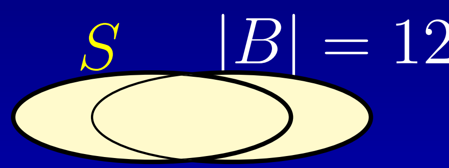

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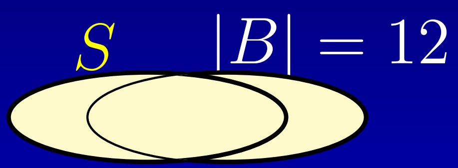

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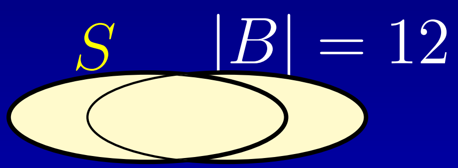

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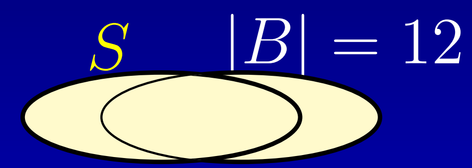

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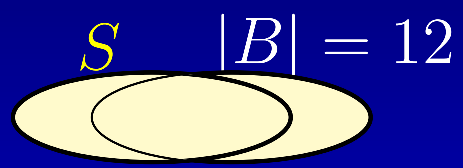

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A quasi-symmetric 2-$(45, 9, 8)$ design is also unique (Harada–M.–Tonchev, 2005).

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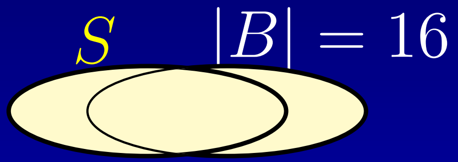

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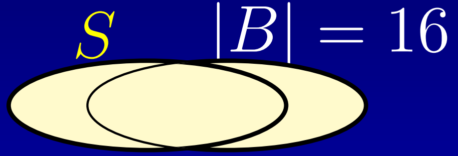

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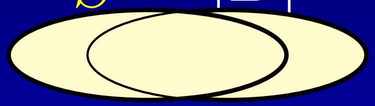

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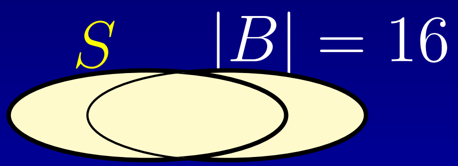

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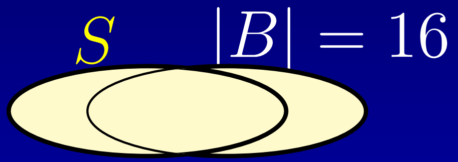

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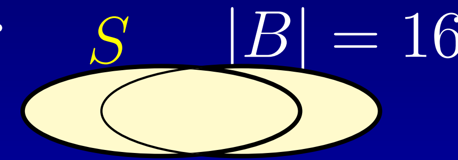

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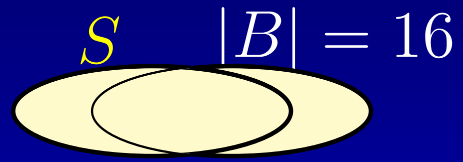

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In particular, $\lambda = 78$.

Spherical analogue

t-design spherical $2t$-design

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t-design	spherical $2t$-design
binary self-orthogonal code	integral lattice
binary self-dual code	unimodular lattice
Assmus–Mattson theorem	Venkov's theorem
extended binary Golay code	Leech lattice
$S(5, 8, 24)$	10-design in $\mathbb{R}^{24}$
extended binary quadratic residue code of length 48	extremal lattice in $\mathbb{R}^{48}$
self-orthogonal 5-$(48, 12, 8)$ design	spherical 10-design in $\mathbb{R}^{48}$
self-orthogonal 5-$(72, 16, 78)$ design	spherical 10-design in $\mathbb{R}^{72}$

Spherical analogue

A spherical t-design X is a finite subset of the sphere $S^{n-1}(\mu) \subset \mathbb{R}^n$ of radius $\sqrt{\mu}$ s.t.

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{\int_{S^{n-1}(\mu)} f dx}{\int_{S^{n-1}(\mu)} 1 dx}$$

holds for any polynomial $f(x)$ of degree $\leq t$.

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$$\frac{1}{b}\sum_{B:\text{ block}} f_T(B) = \frac{\sum_{|B|=k} f_T(B)}{\binom{v}{k}} = \frac{\binom{k}{t}}{\binom{v}{t}}$$

for $\forall$ t-element set T, where

$$f_T(B) = \begin{cases} 1 & \text{if } T \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

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The values $4m+4, 2m+2$ are maximal possible ones.

Codes and lattices satisfying the condition of these theorems are called extremal.

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M. Harada has shown a similar statement for $m = 4$ with an appropriate assumption on the value of λ.

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For $m = 1$, this result implies the characterization of the kissing configuration in $\mathbb{R}^{24}$ by Bannai–Sloane (1981).

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There are infinitely many unknowns, while there are t equations.

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Consistency condition is derived when $t = 10$, $\mu = 4,6,8$ (rank $24,48,72$, respectively).

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Theorem (Bannai–Sloane, 1981). *The set of $196,560$ shortest vectors of the Leech lattice is the unique kissing configuration in $\mathbb{R}^{24}$.*

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- $(t, \mu) = (4, 2) \implies X = E_8$,
- $(t, \mu) = (4, 3) \implies |X| = \dfrac{16n(n+2)}{25-n}$, in particular, $n \leq 24$,

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$$\sum_{j=1}^{[\mu/2]} j^{2i} n_j + 2\mu^{2i} = \frac{(2i-1)!!(\|\alpha\|^2\mu)^i|X|}{n(n+2)\cdots(n+2i-2)} \quad i = 1, \ldots, [t/2].$$

- $(t, \mu) = (4, 2) \implies X = E_8$,
- $(t, \mu) = (4, 3) \implies |X| = \dfrac{16n(n+2)}{25-n}$, in particular, $n \leq 24$,
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Moreover, $[\mu/2] + 1 \leq [t/2] \leq 10$ and $t \leq 10 \implies n$ is bounded.

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A consistency condition is derived when

$$t \geq [k/4] + 1.$$

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which also gives a consistency condition.

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- for each t, k with $t = [k/4] + 1$, v is bounded. Only finitely many (t, k, v)?