An application of Terwilliger's algebra

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Let $\mathcal{X} = (X, \mathcal{R})$ be an association scheme, $\mathcal{R} = \{R_i | 0 \le i \le d\}$. Let $A_0 = I, \ldots, A_d$ be the adjacency matrices of an association scheme. The Terwilliger algebra is by definition the subalgebra of $\operatorname{End} M_n(\mathbf{C})$ generated by the left multiplication by A_i and the Hadamard multiplication by A_i, $0 \le i \le d$. Let e_{xy} $(x, y \in X)$ be matrix unit and take the basis $\{E_{xy,zw}\}$ of $\operatorname{End} M_n(\mathbf{C})$, where $E_{xy,zw} e_{zw} = e_{xy}$. Then the left multiplication by A_i is given by

$$\sum_{x \in X} \sum_{(y,z) \in R_i} E_{yx,zx},$$

while the Hadamard multiplication by A_i is given by

$$\sum_{(y,x) \in R_i} E_{yx,yx}.$$

It follows that the Terwilliger algebra is contained in the subalgebra of $\operatorname{End} M_n(\mathbf{C})$ spanned by $E_{yx,zx}$, $x, y, z \in X$. Since

$$E_{yx,zx} E_{vu,wu} = \delta_{xu} \delta_{zv} E_{yx,wx},$$

we can formally redefine the Terwilliger algebra as a subalgebra of $\mathbf{C}[X \times X \times X]$ with the multiplication

$$(x; y, z)(u; v, w) = \delta_{xu} \delta_{zv} (x; y, w).$$

Namely, by abuse of notation, we can write

$$A_i = \sum_{x \in X} \sum_{(y,z) \in R_i} (x; y, z).$$

If we define

$$E_i^* = \sum_{(x,y) \in R_i} (x; y, y),$$

then the Terwilliger algebra is the subalgebra T of $\mathbf{C}[X \times X \times X]$ generated by A_i, E_i^*, $0 \le i \le d$. First we list some relations among the generators of T. Let $J = \sum_{i=0}^d A_i$, $R_i(x) = \{y \in X | (x, y) \in R_i\}$. We denote by i' the index determined by $R_{i'} = \{(x, y) | (y, x) \in R_i\}$. As in the literature, p_{ij}^k denote the size of the set $R_i(x) \cap R_{j'}(y)$, where $(x, y) \in R_k$.

Lemma 1 (i) $A_0 = \sum_{i=0}^{d} E_i^*$ *is the identity of* T.

(ii) $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$.

(iii) $E_i^* E_j^* = \delta_{ij} E_i^*$.

(iv) $E_i^* A_j E_k^* = \sum_{\substack{(x,y)\in R_i,\ (x,z)\in R_k \\ (y,z)\in R_j}} (x; y, z)$.

(v) $E_0^* A_j = E_0^* A_j E_j^*$, $A_j E_0^* = E_{j'}^* A_j E_0^*$.

(vi) $A_i E_j^* A_k = \sum_{x,y,z\in X} |R_i(y) \cap R_j(x) \cap R_{k'}(z)|(x; y, z)$.

(vii) $A_i E_0^* A_k = E_{i'}^* J E_k^*$.

(viii) $J E_j^* A_k = \sum_{i=0}^{d} p_{jk}^i J E_i^*$.

(ix) $A_i E_j^* J = \sum_{k=0}^{d} p_{ji'}^k E_k^* J$.

(x) $E_0^* A_i E_j^* A_k = \delta_{ij} \sum_{l=0}^{d} p_{ik}^l E_0^* A_l E_l^*$.

(xi) $A_i E_j^* A_k E_0^* = \delta_{jk'} \sum_{l=0}^{d} p_{ik}^l E_{l'}^* A_l E_0^*$.

Proof. Direct calculation. □

Let T_0 be the linear subspace of T spanned by $E_i^* A_j E_k^*$, $(0 \le i, j, k \le d)$. Clearly, T is generated by T_0 as an algebra since T_0 contains A_i and E_i^* for all i, but in general, T_0 may be a proper subspace of T.

Define the Hadamard product by

$$(x; y, z) \circ (u; v, w) = \delta_{xy}\delta_{yu}\delta_{zw}(x; y, z).$$

Lemma 2 (i) J *is the identity with respect to the Hadamard product.*

(ii) $A_l \circ (E_i^*(x; y, z)E_k^*) = E_i^*(A_l \circ (x; y, z))E_k^* = (x; y, z)$ *if* $(x, y) \in R_i$, $(x, z) \in R_k$, $(y, z) \in R_l$, *and* 0 *otherwise.*

(iii) $A_0 \circ (A_i E_j^* A_k) = \delta_{ik'} \sum_{l=0}^{d} p_{jk}^l E_l^* A_0 E_l^*$.

(iv) T_0 *is closed under the Hadamard product.*

Proof. Direct calculation. □

Lemma 3 *The following are equivalent.*

(i) $A_n \circ (E_l^* A_i E_j^* A_k E_m^*) \in T_0$,

(ii) $A_{n'} \circ (E_m^* A_{k'} E_j^* A_{i'} E_l^*) \in T_0$,

(iii) $A_{m'} \circ (E_n^* A_{k'} E_i^* A_{j'} E_{l'}^*) \in T_0$.

Proof. Any one of the above is equivalent to the condition: $|R_i(y) \cap R_j(x) \cap R_{k'}(z)|$ is constant independent of x, y, z with $(x, y) \in R_l$, $(x, z) \in R_m$, $(y, z) \in R_n$. $\square$

Definition. An association scheme $(X, \mathcal{R})$ is called triply regular if the size of the set $R_i(x) \cap R_j(y) \cap R_k(z)$ depends only on (i, j, k, l, m, n), where $(x, y) \in R_l$, $(x, z) \in R_m$, $(y, z) \in R_n$.

Lemma 4 *An association scheme $(X, \mathcal{R})$ is triply regular if and only if $T = T_0$.*

Proof. By definition and Lemma 1 (vi), $(X, \mathcal{R})$ is triply regular if and only if $A_i E_j^* A_k \in T_0$ for any i, j, k. Thus, $T = T_0$ implies triple regularity. Conversely, suppose $A_i E_j^* A_k \in T_0$ for any i, j, k. By Lemma 1 (ii), (iii), any word in A_i, E_j^* $(0 \leq i, j \leq d)$ is a linear combination of words, in which A_i's and E_j^*'s appear alternately. Such a word with more than one A_i's can be rewritten with less number of A_i's since $A_i E_j^* A_k \in T_0$. By induction, we can show that any word in A_i, E_j^* $(0 \leq i, j \leq d)$ is a linear combination of $E_i^* A_j E_k^*$, $E_i^* A_j$, $A_j E_k^*$, A_j, $(0 \leq i, j, k \leq d)$. Since $\sum_{i=0}^{d} E_i^*$ is the identity, all of these belong to T_0, thus $T = T_0$. $\square$

Lemma 5 *Let $\mathcal{X}$ be an association scheme of class 2. If $A_1 E_1^* A_1 \in T_0$, then $\mathcal{X}$ is triply regular.*

Proof. Since $A_0 E_1^* A_1 \in T_0$ and $J E_1^* A_1 \in T_0$ by Lemma 1 (viii), we have $A_2 E_1^* A_1 \in T_0$, and similarly $A_1 E_1^* A_2 \in T_0$, hence $A_2 E_1^* A_2 \in T_0$ also holds. Since $A_0 = E_0^* + E_1^* + E_2^*$, Lemma 1 (vii) implies $A_i E_2^* A_k \in T_0$ for any i, k. Thus $\mathcal{X}$ is triply regular. $\square$

Proposition 6 *Let $\mathcal{X}$ be a symmetric association scheme of class 2. Then $\mathcal{X}$ is triply regular if and only if $\mathcal{R}$ induces an association scheme on subconstituents of $\mathcal{X}$.*

Proof. If $\mathcal{X}$ is triply regular, then clearly $\mathcal{R}$ induces an association scheme on subconstituents of $\mathcal{X}$. Suppose that $\mathcal{R}$ induces an association scheme on subconstituents of $\mathcal{X}$. This is equivalent to

$$E_1^* A_1 E_1^* A_1 E_1^* \in T_0 \quad \text{and} \quad E_2^* A_1 E_1^* A_1 E_2^* \in T_0.$$

By Lemma 2 (iv) we have

$$A_2 \circ (E_1^* A_1 E_1^* A_1 E_1^*) \in T_0 \quad \text{and} \quad A_1 \circ (E_2^* A_1 E_1^* A_1 E_2^*) \in T_0.$$

By Lemma 3 we have

$$A_1 \circ (E_2^* A_1 E_1^* A_1 E_1^*) \in T_0, \quad A_2 \circ (E_2^* A_1 E_1^* A_1 E_1^*) \in T_0.$$

It follows from Lemma 2 (iii) that $E_2^* A_1 E_1^* A_1 E_1^* \in T_0$, and similarly $E_1^* A_1 E_1^* A_1 E_2^* \in T_0$. Now

$$A_1 E_1^* A_1 = (E_0^* + E_1^* + E_2^*) A_1 E_1^* A_1 (E_0^* + E_1^* + E_2^*),$$

so we see that $A_1 E_1^* A_1 \in T_0$ using Lemma 1 (x), (xi). The result follows from Lemma 5. $\square$

Suppose that there exists a spin model defined on the symmetric association scheme $\mathcal{X}$. This means that there exist complex numbers $t_0, t_1, \ldots, t_d$ such that the function $w : X \times X \to \mathbf{C}^\times$ defined by $w(x, y) = t_i$ when $(x, y) \in R_i$, satisfies the following.

(1) $\sum_{y \in X} w(x, y)w(z, y)^{-1} = \delta_{x,z}|X|$, for $x, z \in X$,

(2) $\sum_{x \in X} w(a, x)w(x, b)w(c, x)^{-1} = \sqrt{|X|}w(a, b)w(c, b)^{-1}w(c, a)^{-1}$ for $a, b, c \in X$.

We want to express the equation (2) in the Terwilliger algebra. Put

$$W = \sum_{i=0}^{d} t_i A_i,$$

$$W^* = \sqrt{|X|} \sum_{i=0}^{d} t_i^{-1} E_i^*.$$

Lemma 7 *The equation* (2) *is equivalent to* $WW^*W = W^*WW^*$.

Proof. We have

$$\begin{aligned}
\frac{1}{\sqrt{|X|}} WW^*W &= \sum_{i,j,k} t_i t_j^{-1} t_k A_i E_j^* A_k \\
&= \sum_{i,j,k} \sum_{a,b,c \in X} t_i t_j^{-1} t_k |R_j(c) \cap R_i(a) \cap R_{k'}(b)| (c; a, b) \\
&= \sum_{a,b,c \in X} \sum_{i,j,k} \sum_{x \in R_j(c) \cap R_i(a) \cap R_{k'}(b)} t_i t_j^{-1} t_k (c; a, b) \\
&= \sum_{a,b,c \in X} \sum_{x \in X} w(a, x)w(x, b)w(c, x)^{-1} (c; a, b),
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\sqrt{|X|}} W^*WW^* &= \sqrt{|X|} \sum_{i,j,k} t_i^{-1} t_j t_k^{-1} E_i^* A_j E_k^* \\
&= \sqrt{|X|} \sum_{i,j,k} t_i^{-1} t_j t_k^{-1} \sum_{\substack{(c,a) \in R_i,\ (c,b) \in R_k \\ (a,b) \in R_j}} (c; a, b) \\
&= \sqrt{|X|} \sum_{a,b,c \in X} w(a, b)w(c, b)^{-1} w(c, a)^{-1} (c; a, b).
\end{aligned}$$

Thus the result follows. □

We give a simple proof of a result due to Jaeger.

Theorem 8 *Let $\mathcal{X}$ be symmetric association scheme of class* 2*, $w : X \times X \to \mathbf{C}^\times$ a spin model, $w(x, y) = t_i$ if $(x, y) \in R_i$, and $t_1 \neq t_2$. Then $\mathcal{X}$ is triply regular.*

Proof. Since $W^*WW^* \in T_0$, Lemma 7 implies $WW^*W \in T_0$. By definition, $W = t_0A_0 + t_1A_1 + t_2A_2$ is a linear combination of A_0, A_1 and J. By Lemma 1 (i), (viii), we have $A_0W^*W \in T_0$, $JW^*W \in T_0$. Since $t_1 \neq t_2$, we obtain $A_1W^*W \in T_0$. Similarly we have $A_1W^*A_1 \in T_0$. Moreover, Lemma 1 (vii) implies $A_1E_0^*A_1 \in T_0$, so that we get

$$t_1^{-1}A_1E_1^*A_1 + t_2^{-1}A_1E_2^*A_1 \in T_0.$$

On the other hand,

$$A_1E_1^*A_1 + A_1E_2^*A_1 = A_1^2 - A_1E_0^*A_1 \in T_0.$$

Since $t_1 \neq t_2$, we obtain $A_1E_1^*A_1 \in T_0$. The assertion follows from Lemma 5. □

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