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Definition 6. Let X be a set of formal symbols, and let $F(X)$ be the free group generated by the set of involutions X. Let $R \subset F(X)$. Let N be the subgroup generated by the set

$$\{c^{-1}r^{\pm}c \mid c \in F(X),\ r \in R\}. \tag{21}$$

In other words, N is the set of elements of $F(X)$ expressible as a product of elements in the set (21). The set

$$F(X)/N = \{aN \mid a \in F(X)\},$$

where $aN = \{ab \mid b \in N\}$ for $a \in F(X)$, forms a group under the binary operation

$$\begin{array}{rcl} F(X)/N \times F(X)/N & \to & F(X)/N \\ (aN, bN) & \mapsto & abN \end{array}$$

and it is called the group with presentation $\langle X \mid R \rangle$.

In view of Definition 6, we show that the dihedral group G of order $2m$ is isomorphic to the the group with presentation $\langle x, y \mid (xy)^m \rangle$. Indeed, we have seen that there is a homomorphism $f : F(X) \to G$ with $f(x) = s$ and $f(y) = t$. In our case, $R = \{(xy)^m\}$ which is mapped to 1 under f. So f is constant on each equivalence class, and hence f induces a mapping $\overline{f} : F(X)/N \to G$ defined by $\overline{f}(aN) = f(a)$ $(a \in F(X))$. This mapping $\overline{f}$ is a homomorphism since

$$\begin{aligned} \overline{f}((aN)(bN)) &= \overline{f}(abN) \\ &= f(ab) \\ &= f(a)f(b) \\ &= \overline{f}(aN)\overline{f}(bN). \end{aligned}$$

Moreover, it is clear that both f and $\overline{f}$ are surjective, since $G = \langle s, t \rangle = \langle f(x), f(y) \rangle$. The most important part of the proof is injectivity of $\overline{f}$. The argument on the transformation rule defined by $(xy)^m$ shows

$$F(X)/N = \{(xy)^j N \mid 0 \le j < m\} \cup \{(xy)^j xN \mid 0 \le j < m\}.$$

In particular, $|F(X)/N| \le 2m = |G|$. Since $\overline{f}$ is surjective, equality and injectivity of $\overline{f}$ are forced.

Definition 7. Let V be a finite-dimensional vector space over $\mathbf{R}$ with positive definite inner product. The set $O(V)$ of orthogonal linear transformations of V forms a group under composition. We call $O(V)$ the *orthogonal group* of V.

Definition 8. Let V be a finite-dimensional vector space over $\mathbf{R}$ with positive definite inner product. A subgroup W of the group $O(V)$ is said to be a *finite reflection group* if

(i) $W \neq \{\mathrm{id}_V\}$,

(ii) W is finite,

(iii) W is generated by a set of reflections.

For example, the dihedral group G of order $2m$ is a finite reflection group, since $G \subset O(\mathbf{R}^2)$, $|G| = 2m$ is neither 1 nor infinite, and G is generated by two reflections. We have seen that G has presentation $\langle s, t \mid (st)^m \rangle$. One of the goal of these lectures is to show that every finite reflection group has presentation $\langle s_1, \dots, s_n \mid R \rangle$, where $R \subset F(\{s_1, \dots, s_n\})$ is of the form $\{(s_i s_j)^{m_{ij}} \mid 1 \le i, j \le n\}$.

Let $n \ge 2$ be an integer, and let $\mathcal{S}_n$ denote the symmetric group of degree n. In other words, $\mathcal{S}_n$ consists of all permutations of the set $\{1, 2, \dots, n\}$. Since permutations are bijections from $\{1, 2, \dots, n\}$ to itself, $\mathcal{S}_n$ forms a group under composition. Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of $\mathbf{R}^n$. For each $\sigma \in \mathcal{S}_n$, we define $g_\sigma \in O(\mathbf{R}^n)$ by setting

$$g_\sigma\Big(\sum_{i=1}^{n} c_i \varepsilon_i\Big) = \sum_{i=1}^{n} c_i \varepsilon_{\sigma(i)},$$

and set

$$G_n = \{g_\sigma \mid \sigma \in \mathcal{S}_n\}.$$

It is easy to verify that G_n is a subgroup of $O(V)$ and, the mapping $\mathcal{S}_n \to G_n$ defined by $\sigma \mapsto g_\sigma$ is an isomorphism. We claim that g_σ is a reflection if σ is a transposition; more precisely,

$$g_\sigma = s_{\varepsilon_i - \varepsilon_j} \quad \text{if } \sigma = (i\ j). \tag{22}$$

Indeed, for $k \in \{1, 2, \dots, n\}$,

$$\begin{aligned}
s_{\varepsilon_i - \varepsilon_j}(\varepsilon_k) &= \varepsilon_k - \frac{2(\varepsilon_k, \varepsilon_i - \varepsilon_j)}{(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j)}(\varepsilon_i - \varepsilon_j) \\
&= \varepsilon_k - (\varepsilon_k, \varepsilon_i - \varepsilon_j)(\varepsilon_i - \varepsilon_j) \\
&= \begin{cases} \varepsilon_i - (\varepsilon_i - \varepsilon_j) & \text{if } k = i, \\ \varepsilon_j + (\varepsilon_i - \varepsilon_j) & \text{if } k = j, \\ \varepsilon_k & \text{otherwise} \end{cases} \\
&= \begin{cases} \varepsilon_j & \text{if } k = i, \\ \varepsilon_i & \text{if } k = j, \\ \varepsilon_k & \text{otherwise} \end{cases} \\
&= \varepsilon_{\sigma(k)} \\
&= g_\sigma(\varepsilon_k).
\end{aligned}$$

It is well known that $\mathcal{S}_n$ is generated by its set of transposition. Via the isomorphism $\sigma \mapsto g_\sigma$, we see that G_n is generated by the set of reflections

$$\{s_{\varepsilon_i - \varepsilon_j} \mid 1 \le i < j \le n\}.$$

Therefore, G_n is a finite reflection group.

Observe that G_3 has order 6, and we know another finite reflection group of order 6, namely, the dihedral group of order 6. Although $G_3 \subset O(\mathbf{R}^3)$ while the dihedral group is a subgroup of $O(\mathbf{R}^2)$, these two groups are isomorphic. In order to see their connection, we make a definition.

Definition 9. Let V be a finite-dimensional vector space over $\mathbf{R}$ with positive definite inner product. Let $W \subset O(V)$ be a finite reflection group. We say that W is *not essential* if there exists a nonzero vector $\lambda \in V$ such that $t\lambda = \lambda$ for all $t \in W$. Otherwise, we say that W is *essential*.

For example, the dihedral group G of order $2m \geq 6$ is essential. Indeed, G contains a rotation t whose matrix representation is

$$\begin{bmatrix} \cos\frac{2\pi}{m} & -\sin\frac{2\pi}{m} \\ \sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} \end{bmatrix}. \tag{23}$$

There exists no nonzero vector $\lambda \in V$ such that $t\lambda = \lambda$ since the matrix (23) does not have 1 as an eigenvalue:

$$\begin{vmatrix} \cos\frac{2\pi}{m} - 1 & -\sin\frac{2\pi}{m} \\ \sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} - 1 \end{vmatrix} = 2(1 - \cos\frac{2\pi}{m}) \neq 0.$$

On the other hand, the group G_n which is isomorphic to $\mathcal{S}_n$ is not essential. Indeed, the vector $\lambda = \sum_{i=1}^{n} \varepsilon_i$ is fixed by every $t \in G_n$. In order to find connections between the dihedral group of order 6 and the group G_3, we need a method to produce an essential finite reflection group from non-essential one.

Given a finite reflection group $W \subset O(V)$, let

$$U = \{\lambda \in V \mid \forall t \in W,\ t\lambda = \lambda\}.$$

It is easy to see that U is a subspace of V. Let U' be the orthogonal complement of U in V. Since $tU = U$ for all $t \in W$, we have $tU' = U'$ for all $t \in W$. This allows to construct the restriction homomorphism $W \to O(U')$ defined by $t \mapsto t|_{U'}$.

Exercise 10. Show that the above restriction homomorphism is injective, and the image $W|_{U'}$ is an essential finite reflection group in $O(U')$.

For the group G_3, we have

$$\begin{aligned} U &= \mathbf{R}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \\ U' &= \mathbf{R}(\varepsilon_1 - \varepsilon_2) + \mathbf{R}(\varepsilon_2 - \varepsilon_3) \\ &= \mathbf{R}\eta_1 + \mathbf{R}\eta_2, \end{aligned}$$

where

$$\begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}}(\varepsilon_1 - \varepsilon_2), \\ \eta_2 &= \frac{1}{\sqrt{6}}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3) \end{aligned}$$

is an orthonormal basis of U'.

Exercise 11. Compute the matrix representations of $g_{(1\ 2)}$ and $g_{(2\ 3)}$ with respect to the basis $\{\eta_1, \eta_2\}$. Show that they are reflections whose lines of symmetry form an angle $\pi/3$.

As a consequence of Exercise 10, we see that the group G_3, restricted to the subspace U' so that it becomes essential, is nothing but the dihedral group of order 6.