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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Recall that for $0 \neq \alpha \in V$, $s_\alpha \in O(V)$ denotes the reflection

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad (\lambda \in V). \tag{24}$$

Lemma 12. *For $t \in O(V)$ and $0 \neq \alpha \in V$, we have $ts_\alpha t^{-1} = s_{t\alpha}$.*

Proof. For $\lambda \in V$, we have

$$\begin{aligned} ts_\alpha(\lambda) &= t\left(\lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha\right) && \text{(by (24))} \\ &= t\lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}t\alpha \\ &= t\lambda - \frac{2(t\lambda, t\alpha)}{(t\alpha, t\alpha)}t\alpha \\ &= s_{t\alpha}(t\lambda). \end{aligned}$$

This implies $ts_\alpha = s_{t\alpha}t$, and the result follows. □

For example, if s_α is a reflection in a dihedral group G, and $t \in G$ is a rotation, then s_α and t are not necessarily commutative, but rotating before reflecting can be compensated by reflecting with respect to another line afterwards.

Proposition 13. *Let $W \subset O(V)$ be a finite reflection group, and let $0 \neq \alpha \in V$. If $w, s_\alpha \in W$, then $s_{w\alpha} \in W$.*

Proof. By Lemma 12, we have $s_{w\alpha} = ws_\alpha w^{-1} \in W$. □

Definition 14. Let Φ be a nonempty finite set of nonzero vectors in V. We say that Φ is a *root system* if

(R1) $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,

(R2) $s_\alpha\Phi = \Phi$ for all $\alpha \in \Phi$.

Proposition 15. *Let Φ be a root system in V. Then the subgroup*

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi\rangle$$

of $O(V)$ is a finite reflection group. Moreover, $W(\Phi)$ is essential if and only if Φ spans V. Conversely, for every finite reflection group $W \subset O(V)$, there exists a root system $\Phi \subset V$ such that $W = W(\Phi)$.

Proof. Since $\Phi \neq \emptyset$, the group $W(\Phi)$ contains at least one reflection. In particular, $W(\Phi) \neq \{\mathrm{id}_V\}$. By construction, W is generated by reflections. In order to show that W is finite, let U be the subspace of V spanned by Φ. Since $U^\perp \subset (\mathbf{R}\alpha)^\perp$ for all $\alpha \in \Phi$, we have $s_\alpha(\lambda) = \lambda$ for all $\alpha \in \Phi$ and $\lambda \in U^\perp$. This implies that

$$w|_{U^\perp} = \mathrm{id}_{U^\perp} \quad (w \in W). \tag{25}$$

In particular, W leaves $U^\perp$ invariant. Since $W \subset O(V)$, W also leaves U invariant. We can form the restriction homomorphism $W \to O(U)$ which is injective. Indeed, if an element $w \in W$ is in the kernel of the restriction homomorphism, then $w|_U = \mathrm{id}_U$. Together with (25), we see $w = \mathrm{id}_V$. By (R2), W permutes the finite set Φ, hence there is a homomorphism f from W to the symmetric group on Φ. An element $w \in \operatorname{Ker} f$ fixes every element of Φ, in particular, a basis of U. This implies that w is in the kernel of the restriction homomorphism, and hence $w = \mathrm{id}_V$. We have shown that f is an injection from W to the symmetric group of Φ which is finite. Therefore W is finite. This completes the proof of the first part.

Moreover, $W(\Phi)$ is not essential if and only if there exists a nonzero vector $\lambda \in V$ such that $t\lambda = \lambda$ for all $t \in W(\Phi)$. Since $W(\Phi)$ is generated by $\{s_\alpha \mid \alpha \in \Phi\}$,

$$\begin{aligned} t\lambda = \lambda \ (\forall t \in W(\Phi)) &\iff s_\alpha\lambda = \lambda \ (\forall \alpha \in \Phi) \\ &\iff (\lambda, \alpha) = 0 \ (\forall \alpha \in \Phi) \\ &\iff \lambda \in U^\perp. \end{aligned}$$

Thus, $W(\Phi)$ is not essential if and only if $U^\perp \neq 0$, or equivalently, Φ does not span V.

Conversely, let $W \subset O(V)$ be a finite reflection group, and let S be the set of all reflections of W. By Definition 8(iii), W is generated by S. Define

$$\Phi = \{\alpha \in V \mid s_\alpha \in S,\ \|\alpha\| = 1\}. \tag{26}$$

Observe

$$S = \{s_\alpha \mid \alpha \in \Phi\}. \tag{27}$$

We claim that Φ is a root system. First, since $W \neq \{\mathrm{id}_V\}$, we have $\Phi \neq \emptyset$. Let $\alpha \in \Phi$. Since $s_\alpha = s_{-\alpha}$ and $\|\alpha\| = \|-\alpha\|$, we see that Φ satisfies (R1). For $\beta \in \Phi$, we have $\|s_\alpha(\beta)\| = \|\beta\| = 1$, and $s_{s_\alpha(\beta)} \in W$ by Proposition 13, since $s_\alpha, s_\beta \in W$. This implies $s_\alpha(\beta) \in \Phi$, and hence $s_\alpha(\Phi) = \Phi$. Therefore, Φ is a root system. It remains to show that $W = W(\Phi)$. But this follows immediately from (27) since $W = \langle S \rangle$. □

Example 16. We have seen that the group G_n generated by reflections

$$\{s_{\varepsilon_i - \varepsilon_j} \mid 1 \leq i < j \leq n\}, \tag{28}$$

where $\varepsilon_1, \dots, \varepsilon_n$ is the standard basis of $\mathbf{R}^n$, is a finite reflection group which is abstractly isomorphic to the symmetric group of degree n. The set

$$\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\} \tag{29}$$

is a root system. Indeed, Φ clearly satisfies (R1). It is also clear that $g_\sigma\Phi = \Phi$ for all $\sigma \in \mathcal{S}_n$, so in particular, (R2) holds.

Exercise 17. Show that (28) is precisely the set of reflections in G_n. In other words, show that g_σ is a reflection if and only if σ is a transposition.

Definition 18. A *total ordering* of V is a transitive relation on V (denoted $<$) satisfying the following axioms.

(i) For each pair $\lambda, \mu \in V$, exactly one of $\lambda < \mu$, $\lambda = \mu$, $\mu < \lambda$ holds.

(ii) For all $\lambda, \mu, \nu \in V$, $\mu < \nu$ implies $\lambda + \mu < \lambda + \nu$.

(iii) Let $\mu < \nu$ and $c \in \mathbf{R}$. If $c > 0$ then $c\mu < c\nu$, and if $c < 0$ then $c\nu < c\mu$.

For convenience, we write $\lambda > \mu$ if $\mu < \lambda$. By (ii), $\lambda > 0$ implies $0 > -\lambda$. Thus

$$V = V_+ \cup \{0\} \cup V_- \quad (\text{disjoint}), \tag{30}$$

where

$$V_+ = \{\lambda \in V \mid \lambda > 0\}, \tag{31}$$
$$V_- = \{\lambda \in V \mid \lambda < 0\}. \tag{32}$$

We say that $\lambda \in V_+$ is *positive*, and $\lambda \in V_-$ is *negative*.

Example 19. Let $\lambda_1, \dots, \lambda_n$ be a basis of V. Define the lexicographic ordering of V with respect to $\lambda_1, \dots, \lambda_n$ by

$$\sum_{i=1}^{n} a_i\lambda_i < \sum_{i=1}^{n} b_i\lambda_i \iff \exists k \in \{1, 2, \dots, n\},\ a_1 = b_1, \dots, a_{k-1} = b_{k-1}, a_k < b_k.$$

Clearly, this is a total ordering of V. Note that $\lambda_i > 0$ for all $i \in \{1, \dots, n\}$. For $n = 2$, we have

$$V_+ = \{c_1\lambda_1 + c_2\lambda_2 \mid c_1 > 0,\ c_2 \in \mathbf{R}\} \cup \{c_2\lambda_2 \mid c_2 > 0\}.$$

Lemma 20. *Let $<$ be a total ordering of V, and let $\lambda, \mu \in V$.*

(i) *If $\lambda, \mu > 0$, then $\lambda + \mu > 0$.*

(ii) *If $\lambda > 0$, $c \in \mathbf{R}$ and $c > 0$, then $c\lambda > 0$.*

(iii) *If $\lambda > 0$, $c \in \mathbf{R}$ and $c < 0$, then $c\lambda < 0$. In particular, $-\lambda < 0$.*

Proof. (i) By Definition 18(ii), we have $\lambda + \mu > \lambda > 0$.

(ii) By Definition 18(iii), we have $c\lambda > c \cdot 0 = 0$.

(iii) By Definition 18(iii), we have $c\lambda < c \cdot 0 = 0$. Taking $c = -1$ gives the second statement. □

Definition 21. Let Φ be a root system in V. A subset Π of Φ is called a *positive system* if there exists a total ordering $<$ of V such that

$$\Pi = \{\alpha \in \Phi \mid \alpha > 0\}. \tag{33}$$

Since a total ordering of V always exists by Example 19, and every total ordering of V defines a positive system of a root system Φ in V, according to Definition 21, there are many positive systems in Φ.

Example 22. Continuing Example 16, let $<$ be the total ordering defined by the basis $\varepsilon_1, \dots, \varepsilon_n$. Then $\varepsilon_i > \varepsilon_j$ if $i < j$. Thus, according to (33),

$$\Pi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}.$$

Lemma 23. *If* Π *is a positive system in a root system* Φ*, then* $\Phi = \Pi \cup (-\Pi)$ *(disjoint), where*

$$-\Pi = \{-\alpha \mid \alpha \in \Pi\}. \tag{34}$$

In particular,

$$-\Pi = \{\alpha \in \Phi \mid \alpha < 0\}. \tag{35}$$

Proof. We have

$$\begin{aligned} \Pi \cap (-\Pi) &= \emptyset && \text{(by Lemma 20(iii))}, \\ \Pi &\subset \Phi && \text{(by Definition 21)}, \\ -\Pi &\subset \Phi && \text{(by Definition 14(R1))}. \end{aligned}$$

Thus, it remains to show $\Phi \subset \Pi \cup (-\Pi)$. Suppose $\alpha \in \Phi \setminus \Pi$. Then

$$\begin{aligned} \alpha \notin \Pi &\implies \alpha \not> 0 && \text{(by (33))} \\ &\implies \alpha < 0 && \text{(since } 0 \notin \Phi) \\ &\implies 0 < -\alpha && \text{(by Definition 18(ii)} \\ &\implies -\alpha \in \Pi && \text{(by (33))} \\ &\implies \alpha \in -\Pi && \text{(by (34))}. \end{aligned}$$

This proves $\Phi \setminus \Pi \subset (-\Pi)$, proving $\Phi \subset \Pi \cup (-\Pi)$.

Since $\Phi = \Pi \cup (-\Pi)$ (disjoint) and $0 \notin \Phi$, (33) implies (35). □

Definition 24. Let Π be a positive system in a root system Φ. We call $-\Pi$ defined by (34) the *negative system* in Φ with respect to Π.

Definition 25. Let Δ be a subset of a root system Φ. We call Δ a *simple system* if Δ is a basis of the subspace spanned by Φ, and if moreover each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign (all nonnegative or all nonpositive). In other words,

$$\Phi \subset \mathbf{R}_{\ge 0}\Delta \cup \mathbf{R}_{\le 0}\Delta, \tag{36}$$

where

$$\mathbf{R}_{\ge 0}\Delta = \{\sum_{\alpha \in \Delta} c_\alpha \alpha \mid c_\alpha \ge 0 \ (\alpha \in \Delta)\}.$$

If Δ is a simple system, we call its elements *simple roots*.

Example 26. Continuing Example 22,

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < n\} \tag{37}$$

is a simple system. Indeed, for $\varepsilon_i - \varepsilon_j \in \Phi$, we have

$$\varepsilon_i - \varepsilon_j = \begin{cases} \sum_{k=i}^{j-1} (\varepsilon_k - \varepsilon_{k+1}) \in \mathbf{R}_{\geq 0}\Delta & \text{if } i < j, \\ \sum_{k=j}^{i-1} (-(\varepsilon_j - \varepsilon_{j+1})) \in \mathbf{R}_{\leq 0}\Delta & \text{otherwise.} \end{cases}$$