Definition 1. Let V be a finite-dimensional vector space over $\mathbf{R}$ with positive definite inner product. The set $O(V)$ of orthogonal linear transformations of V forms a group under composition. We call $O(V)$ the *orthogonal group* of V.

Definition 2. Let V be a finite-dimensional vector space over $\mathbf{R}$ with positive definite inner product. A subgroup W of the group $O(V)$ is said to be a *finite reflection group* if

(i) $W \neq \{\mathrm{id}_V\}$,

(ii) W is finite,

(iii) W is generated by a set of reflections.

Let $n \geq 2$ be an integer, and let $\mathcal{S}_n$ denote the symmetric group of degree n. In other words, $\mathcal{S}_n$ consists of all permutations of the set $\{1, 2, \dots, n\}$. Since permutations are bijections from $\{1, 2, \dots, n\}$ to itself, $\mathcal{S}_n$ forms a group under composition. Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of $\mathbf{R}^n$. For each $\sigma \in \mathcal{S}_n$, we define $g_\sigma \in O(\mathbf{R}^n)$ by setting

$$g_\sigma(\sum_{i=1}^{n} c_i \varepsilon_i) = \sum_{i=1}^{n} c_i \varepsilon_{\sigma(i)},$$

and set

$$G_n = \{g_\sigma \mid \sigma \in \mathcal{S}_n\}.$$

It is easy to verify that G_n is a subgroup of $O(V)$ and, the mapping $\mathcal{S}_n \to G_n$ defined by $\sigma \mapsto g_\sigma$ is an isomorphism. We claim that g_σ is a reflection if σ is a transposition; more precisely,

$$g_\sigma = s_{\varepsilon_i - \varepsilon_j} \quad \text{if } \sigma = (i\ j). \tag{1}$$

It is well known that $\mathcal{S}_n$ is generated by its set of transposition. Via the isomorphism $\sigma \mapsto g_\sigma$, we see that G_n is generated by the set of reflections

$$\{s_{\varepsilon_i - \varepsilon_j} \mid 1 \leq i < j \leq n\}.$$

Therefore, G_n is a finite reflection group.

Definition 3. Let V be a finite-dimensional vector space over $\mathbf{R}$ with positive definite inner product. Let $W \subset O(V)$ be a finite reflection group. We say that W is *not essential* if there exists a nonzero vector $\lambda \in V$ such that $t\lambda = \lambda$ for all $t \in W$. Otherwise, we say that W is *essential*.

Given a finite reflection group $W \subset O(V)$, let

$$U = \{\lambda \in V \mid \forall t \in W,\ t\lambda = \lambda\}.$$

It is easy to see that U is a subspace of V. Let U' be the orthogonal complement of U in V. Since $tU = U$ for all $t \in W$, we have $tU' = U'$ for all $t \in W$. This allows to construct the restriction homomorphism $W \to O(U')$ defined by $t \mapsto t|_{U'}$.

Exercise 4. Show that the above restriction homomorphism is injective, and the image $W|_{U'}$ is an essential finite reflection group in $O(U')$.

For the group G_3, we have

$$\begin{aligned} U &= \mathbf{R}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \\ U' &= \mathbf{R}(\varepsilon_1 - \varepsilon_2) + \mathbf{R}(\varepsilon_2 - \varepsilon_3) \\ &= \mathbf{R}\eta_1 + \mathbf{R}\eta_2, \end{aligned}$$

where

$$\begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}}(\varepsilon_1 - \varepsilon_2), \\ \eta_2 &= \frac{1}{\sqrt{6}}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3) \end{aligned}$$

is an orthonormal basis of U'.

Exercise 5. Compute the matrix representations of $g_{(1\ 2)}$ and $g_{(2\ 3)}$ with respect to the basis $\{\eta_1, \eta_2\}$. Show that they are reflections whose lines of symmetry form an angle $\pi/3$.