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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product.

Recall that a total ordering $<$ of V partitions V into three parts

$$V = V_+ \cup \{0\} \cup (-V_+),$$

such that

$$V_+ + V_+ \subset V_+, \tag{38}$$

$$\mathbf{R}_{\geq 0} V_+ \subset V_+ \cup \{0\}. \tag{39}$$

Lemma 27. *Let Δ be a finite set of nonzero vectors in V_+. If $(\alpha, \beta) \leq 0$ for any distinct $\alpha, \beta \in \Delta$, then Δ consists of linearly independent vectors.*

Proof. Let

$$\sum_{\alpha \in \Delta} a_\alpha \alpha = 0, \tag{40}$$

and define

$$\sigma = \sum_{\substack{\alpha \in \Delta \\ a_\alpha > 0}} a_\alpha \alpha.$$

Then

$$\begin{aligned} 0 &\leq (\sigma, \sigma) \\ &= \Big(\sum_{\substack{\alpha \in \Delta \\ a_\alpha > 0}} a_\alpha \alpha, \sum_{\alpha \in \Delta} a_\alpha \alpha - \sum_{\substack{\beta \in \Delta \\ a_\beta < 0}} a_\beta \beta\Big) \\ &= \Big(\sum_{\substack{\alpha \in \Delta \\ a_\alpha > 0}} a_\alpha \alpha, -\sum_{\substack{\beta \in \Delta \\ a_\beta < 0}} a_\beta \beta\Big) && \text{(by (40))} \\ &= -\sum_{\substack{\alpha \in \Delta \\ a_\alpha > 0}} \sum_{\substack{\beta \in \Delta \\ a_\beta < 0}} a_\alpha a_\beta (\alpha, \beta) \\ &\leq 0. \end{aligned}$$

This forces $\sigma = 0$, so there is no $\alpha \in \Delta$ with $a_\alpha > 0$. Similarly, we can show that there is no $\alpha \in \Delta$ with $a_\alpha < 0$. Therefore, $a_\alpha = 0$ for all $\alpha \in \Delta$. □

Lemma 28. *Let $\Delta \subset V_+$ be a subset, and let $\alpha, \beta \in \Delta$ be linearly independent. If $\alpha \in \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}\Delta$, then $\alpha \in \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\})$.*

Proof. Since

$$\alpha \in \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}\Delta$$

$$
\begin{aligned}
&= \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}\alpha + \mathbf{R}_{\geq 0}\beta + \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha, \beta\}) \\
&= \mathbf{R}_{\geq 0}\alpha + \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha, \beta\}) \\
&\subset \mathbf{R}_{\geq 0}\alpha + V_+ \cap \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\}),
\end{aligned}
$$

there exists $a \in \mathbf{R}_{\geq 0}$ such that

$$\alpha \in a\alpha + V_+ \cap \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\}). \tag{41}$$

Thus

$$(1-a)\alpha \in V_+, \tag{42}$$
$$(1-a)\alpha \in \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\}). \tag{43}$$

By (42), we have $1 - a > 0$. The result then follows from (43). □

For a root system Φ in V, we denote by $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$, the set of positive systems and that of simple systems, respectively, in Φ. More specifically,

$$
\begin{aligned}
\mathcal{P}(\Phi) &= \{\{\alpha \in \Phi \mid \alpha > 0\} \mid \text{“}>\text{” is a total ordering of } V\}, \\
\mathcal{S}(\Phi) &= \{\Delta \subset \Phi \mid \Phi \subset \mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta,\ \Delta \text{ is linearly independent}\}.
\end{aligned}
$$

It is clear that $\mathcal{P}(\Phi)$ is non-empty, since V can be given a total ordering. We show that $\mathcal{S}(\Phi)$ is non-empty by establishing a bijection between $\mathcal{S}(\Phi)$ and $\mathcal{P}(\Phi)$, which is defined by

$$
\begin{array}{rcl}
\pi : \mathcal{S}(\Phi) & \to & \mathcal{P}(\Phi) \\
\Delta & \mapsto & \Phi \cap \mathbf{R}_{\geq 0}\Delta.
\end{array} \tag{44}
$$

Lemma 29. *Let Φ be a root system in V. If Δ is a simple system contained in a positive system Π, then*

(i) $\Pi = \Phi \cap \mathbf{R}_{\geq 0}\Delta$,

(ii) $\Delta = \{\alpha \in \Pi \mid \alpha \notin \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})\}$.

Proof. (i) Since Δ is a simple system, we have

$$\Phi \subset \mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta. \tag{45}$$

Since $\Delta \subset \Pi \subset V_+$ for some total ordering of V, we have

$$\mathbf{R}_{\geq 0}\Delta \subset V_+ \cup \{0\}, \tag{46}$$
$$\mathbf{R}_{\leq 0}\Delta \subset V_- \cup \{0\}. \tag{47}$$

Thus

$$
\begin{aligned}
\Pi &= \Phi \cap V_+ \\
&= \Phi \cap (\mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta) \cap V_+ && \text{(by (45))}
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi \cap \mathbf{R}_{\geq 0}\Delta \cap V_+ && \text{(by (47))}\\
&= \Phi \cap (\mathbf{R}_{\geq 0}\Delta \setminus \{0\}) && \text{(by (46))}\\
&= \Phi \cap \mathbf{R}_{\geq 0}\Delta.
\end{aligned}
$$

(ii) If $\alpha \in \Pi \setminus \Delta$, then $\Delta \subset \Pi \setminus \{\alpha\}$, so $\mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\}) \supset \mathbf{R}_{\geq 0}\Delta \ni \alpha$. This proves

$$\Delta \supset \{\alpha \in \Pi \mid \alpha \notin \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})\}.$$

Conversely, suppose $\alpha \in \Pi$ and $\alpha \in \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})$. Then there exists $\beta \in \Pi \setminus \{\alpha\}$ such that

$$
\begin{aligned}
\alpha &\in \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha, \beta\})\\
&\subset \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}\Pi\\
&= \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}\Delta && \text{(by (i)).}
\end{aligned}
$$

Since $\beta \in \Pi \setminus \{\alpha\} \subset \mathbf{R}_{\geq 0}\Delta \setminus \mathbf{R}_{\geq 0}\alpha$, there exists $\delta \in \Delta \setminus \{\alpha\}$ such that

$$\beta \in \mathbf{R}_{>0}\delta + \mathbf{R}_{\geq 0}\Delta.$$

Thus $\alpha \in \mathbf{R}_{>0}\delta + \mathbf{R}_{\geq 0}\Delta$, and hence $\{\alpha\} \cup \Delta$ is linearly dependent. This implies $\alpha \notin \Delta$. $\square$

Recall that for $0 \neq \alpha \in V$, $s_\alpha \in O(V)$ denotes the reflection

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad (\lambda \in V). \tag{48}$$

Theorem 30. *Let Φ be a root system in V. Then the mapping $\pi : \mathcal{S}(\Phi) \to \mathcal{P}(\Phi)$ defined by* (44) *is a bijection whose inverse is given by*

$$
\begin{aligned}
\pi^{-1} : \mathcal{P}(\Phi) &\to \mathcal{S}(\Phi)\\
\Pi &\mapsto \{\alpha \in \Pi \mid \alpha \notin \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})\}.
\end{aligned} \tag{49}
$$

Moreover,

(i) *for every simple system Δ in Φ, $\pi(\Delta)$ is the unique positive system containing Δ,*

(ii) *for every positive system Π in Φ, $\pi^{-1}(\Pi)$ is the unique simple system contained in* Π.

Proof. If $\Delta \in \mathcal{S}(\Phi)$, then Δ is a basis of the subspace spanned by Φ, so there exists a basis $\tilde{\Delta}$ of V containing Δ. By Example 19, there exists a total ordering $<$ of V such that $\alpha > 0$ for all $\alpha \in \tilde{\Delta}$. Then

$$
\begin{aligned}
\pi(\Delta) &= \Phi \cap \mathbf{R}_{\geq 0}\Delta\\
&= \Phi \cap (\mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta) \cap V_+\\
&= \Phi \cap V_+
\end{aligned}
$$

is a positive system containing Δ.

Next we show that π is injective. Suppose $\Delta, \Delta' \in \mathcal{S}(\Phi)$ and $\pi(\Delta) = \pi(\Delta')$. Then both Δ and Δ' are simple system contained in $\Pi = \pi(\Delta)$. By Lemma 29(ii), we have

$$\Delta = \{\alpha \in \Pi \mid \alpha \notin \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})\} = \Delta'.$$

Therefore, π is injective. Note that this shows

$$\pi^{-1}(\Pi) \subset \{\{\alpha \in \Pi \mid \alpha \notin \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})\}\}. \tag{50}$$

Next we show that π is surjective. Suppose $\Pi \in \mathcal{P}(\Phi)$. Define $\mathcal{D}$ by

$$\mathcal{D} = \{\Delta \subset \Pi \mid \Pi \subset \mathbf{R}_{\geq 0}\Delta\}. \tag{51}$$

Since Φ is a finite set, so are Π and $\mathcal{D}$. Since $\Pi \in \mathcal{D}$, $\mathcal{D}$ is non-empty. Thus, there exists a minimal member Δ of $\mathcal{D}$. This means

$$\Pi \subset \mathbf{R}_{\geq 0}\Delta, \tag{52}$$

$$\forall \alpha \in \Delta,\ \Pi \not\subset \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\}). \tag{53}$$

Since Π is a positive system, there exists a total ordering of V such that $\Pi = \Phi \cap V_+$. In particular, $\Delta \subset V_+$. We claim

$$(\alpha, \beta) \leq 0 \text{ for all pairs } \alpha \neq \beta \text{ in } \Delta. \tag{54}$$

Indeed, suppose, to the contrary, $(\alpha, \beta) > 0$ for some distinct $\alpha, \beta \in \Delta$. Since $\pm s_\alpha(\beta) \in \Phi = \Pi \cup (-\Pi)$, in view of (48), we may assume without loss of generality $\alpha \in \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}\Delta$. Then by Lemma 28, we obtain $\alpha \in \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\})$. Now

$$\begin{aligned} \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\}) &= \mathbf{R}_{\geq 0}\alpha + \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\}) \\ &= \mathbf{R}_{\geq 0}\Delta \\ &\supset \Pi, \end{aligned}$$

contradicting (53). This proves (54). Now, by Lemma 27, Δ consists of linearly independent vectors. We have shown that Δ is a simple system, and by construction, $\Delta \subset \Pi$. Lemma 29(i) then implies $\Pi = \pi(\Delta)$. Therefore, π is surjective. This also implies that equality holds in (50), which shows that the inverse π^{-1} is given by (49).

Finally, (i) follows from Lemma 29(i), while (ii) follows from Lemma 29(ii). □