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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product.

Recall that for $0 \neq \alpha \in V$, $s_\alpha \in O(V)$ denotes the reflection

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad (\lambda \in V).$$

Definition 1. Let Φ be a nonempty finite set of nonzero vectors in V. We say that Φ is a *root system* if

(R1) $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,

(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Definition 2. A *total ordering* of V is a transitive relation on V (denoted $<$) satisfying the following axioms.

(i) For each pair $\lambda, \mu \in V$, exactly one of $\lambda < \mu$, $\lambda = \mu$, $\mu < \lambda$ holds.

(ii) For all $\lambda, \mu, \nu \in V$, $\mu < \nu$ implies $\lambda + \mu < \lambda + \nu$.

(iii) Let $\mu < \nu$ and $c \in \mathbf{R}$. If $c > 0$ then $c\mu < c\nu$, and if $c < 0$ then $c\nu < c\mu$.

For convenience, we write $\lambda > \mu$ if $\mu < \lambda$. By (ii), $\lambda > 0$ implies $0 > -\lambda$. Thus

$$V = V_+ \cup \{0\} \cup V_- \quad \text{(disjoint)},$$

where

$$V_+ = \{\lambda \in V \mid \lambda > 0\},$$
$$V_- = \{\lambda \in V \mid \lambda < 0\}.$$

We say that $\lambda \in V_+$ is *positive*, and $\lambda \in V_-$ is *negative*.

Example 3. Let $\lambda_1, \dots, \lambda_n$ be a basis of V. Define the lexicographic ordering of V with respect to $\lambda_1, \dots, \lambda_n$ by

$$\sum_{i=1}^n a_i\lambda_i < \sum_{i=1}^n b_i\lambda_i \iff \exists k \in \{1, 2, \dots, n\},\ a_1 = b_1, \dots, a_{k-1} = b_{k-1}, a_k < b_k.$$

Clearly, this is a total ordering of V. Note that $\lambda_i > 0$ for all $i \in \{1, \dots, n\}$.

Lemma 4. *Let $<$ be a total ordering of V, and let $\lambda, \mu \in V$.*

(i) *If $\lambda, \mu > 0$, then $\lambda + \mu > 0$.*

(ii) *If $\lambda > 0$, $c \in \mathbf{R}$ and $c > 0$, then $c\lambda > 0$.*

(iii) *If* $\lambda > 0$, $c \in \mathbf{R}$ *and* $c < 0$, *then* $c\lambda < 0$. *In particular,* $-\lambda < 0$.

Definition 5. Let Φ be a root system in V. A subset Π of Φ is called a *positive system* if there exists a total ordering $<$ of V such that

$$\Pi = \{\alpha \in \Phi \mid \alpha > 0\}.$$

Definition 6. Let Δ be a subset of a root system Φ. We call Δ a *simple system* if Δ is a basis of the subspace spanned by Φ, and if moreover each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign (all nonnegative or all nonpositive). In other words,

$$\Phi \subset \mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta,$$

where

$$\mathbf{R}_{\geq 0}\Delta = \{\sum_{\alpha\in\Delta} c_\alpha \alpha \mid c_\alpha \geq 0 \ (\alpha \in \Delta)\}.$$

If Δ is a simple system, we call its elements *simple roots*.

Example 7. Let $n \geq 2$ be an integer, and let $\mathcal{S}_n$ denote the symmetric group of degree n. In other words, $\mathcal{S}_n$ consists of all permutations of the set $\{1, 2, \dots, n\}$. Since permutations are bijections from $\{1, 2, \dots, n\}$ to itself, $\mathcal{S}_n$ forms a group under composition. Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of $\mathbf{R}^n$. For each $\sigma \in \mathcal{S}_n$, we define $g_\sigma \in O(\mathbf{R}^n)$ by setting

$$g_\sigma(\sum_{i=1}^n c_i\varepsilon_i) = \sum_{i=1}^n c_i\varepsilon_{\sigma(i)},$$

and set

$$G_n = \{g_\sigma \mid \sigma \in \mathcal{S}_n\}.$$

It is easy to verify that G_n is a subgroup of $O(V)$ and, the mapping $\mathcal{S}_n \to G_n$ defined by $\sigma \mapsto g_\sigma$ is an isomorphism.

It is well known that $\mathcal{S}_n$ is generated by its set of transposition. Via the isomorphism $\sigma \mapsto g_\sigma$, we see that G_n is generated by the set of reflections

$$\{s_{\varepsilon_i - \varepsilon_j} \mid 1 \leq i < j \leq n\}. \qquad (1)$$

Exercise 8. Show that (1) is precisely the set of reflections in G_n. In other words, show that g_σ is a reflection if and only if σ is a transposition.