May 30, 2016

For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. We also let Φ be a root system in V. Recall that $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$ denote the set of positive systems and that of simple systems, respectively, in Φ. Define

$$\begin{array}{rcl} \pi : \mathcal{S}(\Phi) & \to & \mathcal{P}(\Phi) \\ \Delta & \mapsto & \Phi \cap \mathbf{R}_{\geq 0}\Delta. \end{array}$$

Theorem 30 is proved in an awkward manner, in the sense that $\pi^{-1}(\Pi) \in \mathcal{S}(\Phi)$ for $\Pi \in \mathcal{P}(\Phi)$ is not explicitly shown. Lemma 29(ii) shows that the existence of a simple system in Π does imply $\pi^{-1}(\Pi) \in \mathcal{S}(\Phi)$, but showing the existence of a simple system in Π is a separate problem. Here is how one can show $\pi^{-1}(\Pi) \in \mathcal{S}(\Phi)$ directly. We need a lemma.

Lemma 31. *Suppose that V is given a total ordering, let $A \subset V_+$ be a subset, $\alpha_1, \dots, \alpha_n \in V_+$, and $\beta \in V_+ \setminus \bigcup_{i=1}^n \mathbf{R}\alpha_i$. If*

$$\alpha_i \in \mathbf{R}_{\geq 0}(A \cup \{\beta\}), \tag{55}$$

$$\beta \in \mathbf{R}_{\geq 0}(A \cup \{\alpha_1, \dots, \alpha_n\}), \tag{56}$$

then $\alpha_1, \dots, \alpha_n, \beta \in \mathbf{R}_{\geq 0}A$.

Proof. Let $\mathcal{A} = \mathbf{R}_{\geq 0}A$, $\mathcal{A}_+ = \mathcal{A} \setminus \{0\}$. By the assumption, we have $\mathcal{A}_+ \subset V_+$. Then it suffices to show

$$\beta \in \mathcal{A} \tag{57}$$

only, since $\alpha_i \in \mathcal{A}$ follows immediately from (55) and (57).

By (55), there exist $b_i \in \mathbf{R}_{\geq 0}$ and $\lambda_i \in \mathcal{A}$ such that

$$\alpha_i = b_i\beta + \lambda_i. \tag{58}$$

Since $\beta \notin \mathbf{R}\alpha_i$, we have $\lambda_i \neq 0$, i.e.,

$$\lambda_i \in \mathcal{A}_+. \tag{59}$$

By (56), there exist $a_1, \dots, a_n \in \mathbf{R}_{\geq 0}$ such that

$$\beta \in \sum_{i=1}^n a_i\alpha_i + \mathcal{A}. \tag{60}$$

If $a_i = 0$ for all i, then (57) holds, so we may assume $a_i > 0$ for some i. Then (59) implies

$$\sum_{i=1}^n a_i\lambda_i \in \mathcal{A}_+. \tag{61}$$

By (58) and (60), we obtain

$$\beta \in \sum_{i=1}^n a_i(b_i\beta + \lambda_i) + \mathcal{A}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} a_i b_i \beta + \sum_{i=1}^{n} a_i \lambda_i + \mathcal{A} \\
&\subset \sum_{i=1}^{n} a_i b_i \beta + \mathcal{A}_+ && \text{(by (61))} \\
&= \sum_{i=1}^{n} a_i b_i \beta + V_+ \cap \mathcal{A}.
\end{aligned}
$$

This implies

$$\left(1 - \sum_{i=1}^{n} a_i b_i\right) \beta \in V_+, \tag{62}$$

$$\left(1 - \sum_{i=1}^{n} a_i b_i\right) \beta \in \mathcal{A}. \tag{63}$$

By (62), we have $1 - \sum_{i=1}^{n} a_i b_i > 0$. Then (57) follows from (63). □

Proposition 32. *Let* $\Pi \in \mathcal{P}(\Phi)$*, and set*

$$\Delta = \{\alpha \in \Pi \mid \alpha \notin \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})\}.$$

Then

(i) $(\alpha, \beta) \leq 0$ *for all* $\alpha \neq \beta$ *in* Δ,

(ii) Δ *is a simple system in* Φ.

Proof. (i) Suppose, to the contrary, $(\alpha, \beta) > 0$ for some distinct $\alpha, \beta \in \Delta$. Since $\pm s_\alpha(\beta) \in \Phi = \Pi \cup (-\Pi)$, in view of (48), we may assume without loss of generality $\alpha \in \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}\Pi$. By Lemma 28, we obtain $\alpha \in \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})$, which contradicts $\alpha \in \Delta$.

(ii) By (i) and Lemma 27, Δ consists of linearly independent vectors. It remains to show $\Pi \subset \mathbf{R}_{\geq 0}\Delta$. We consider the set

$$\mathcal{B} = \{B \subset \Pi \setminus \Delta \mid B \subset \mathbf{R}_{\geq 0}(\Pi \setminus B)\}.$$

For all $\alpha \in \Pi \setminus \Delta$, we have $\alpha \in \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})$. Thus $\{\alpha\} \in \mathcal{B}$, and hence $\mathcal{B} \neq \emptyset$.

Let $B = \{\alpha_1, \dots, \alpha_n\}$ be a maximal member of $\mathcal{B}$. Suppose $B \subsetneq \Pi \setminus \Delta$. Then there exists $\beta \in \Pi \setminus (B \cup \Delta)$. Set $A = \Pi \setminus (B \cup \{\beta\})$. Then (55) holds since $B \in \mathcal{B}$, while (56) holds since $\beta \notin \Delta$. Lemma 31 then implies $\alpha_1, \dots, \alpha_n, \beta \in \mathbf{R}_{\geq 0}(\Pi \setminus (B \cup \{\beta\})$. This implies $B \cup \{\beta\} \in \mathcal{B}$, contradicting maximality of B. Therefore, $B = \Pi \setminus \Delta$. This implies $\Pi \setminus \Delta \in \mathcal{B}$, which in turn implies $\Pi \setminus \Delta \subset \mathbf{R}_{\geq 0}\Delta$. Since $\Delta \subset \mathbf{R}_{\geq 0}\Delta$ holds trivially, we obtain $\Pi \subset \mathbf{R}_{\geq 0}\Delta$. This completes the proof of (ii). □

Recall

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle.$$

By Definition 14(R2), we have

$$w\Phi = \Phi \quad (w \in W(\Phi)). \tag{64}$$

Lemma 33. *Let $w \in W(\Phi)$. Then*

(i) $w\Delta \in \mathcal{S}(\Phi)$ *and* $\pi(w\Delta) = w\pi(\Delta)$ *for all* $\Delta \in \mathcal{S}(\Phi)$,

(ii) $w\Pi \in \mathcal{P}(\Phi)$ *and* $\pi^{-1}(w\Pi) = w\pi^{-1}(\Pi)$ *for all* $\Pi \in \mathcal{P}(\Phi)$.

Proof. (i) Clear from (64) and (44).

(ii) For $\Pi \in \mathcal{P}(\Phi)$, let $\Delta = \pi^{-1}(\Pi) \in \mathcal{S}(\Phi)$. Then $w\Pi = w\pi(\Delta) = \pi(w\Delta) \in \pi(\mathcal{S}(\Phi)) = \mathcal{P}(\Phi)$ by (i). Also, $\pi^{-1}(w\Pi) = w\Delta = w\pi^{-1}(\Pi)$. □

Lemma 34. *Let $\alpha \in \Delta \in \mathcal{S}(\Phi)$ and $\Pi = \pi(\Delta)$. Then $s_\alpha(\Pi \setminus \{\alpha\}) = \Pi \setminus \{\alpha\}$.*

Proof. Let $\beta \in \Pi \setminus \{\alpha\}$, and write $\beta = \sum_{\gamma\in\Delta} c_\gamma \gamma$. Then

$$\exists \gamma \in \Delta \setminus \{\alpha\},\ c_\gamma > 0. \tag{65}$$

Set

$$c = \frac{2(\beta, \alpha)}{(\alpha, \alpha)},$$

so that

$$\begin{aligned} s_\alpha\beta &= \beta - c\alpha \\ &= \sum_{\gamma\in\Delta} c_\gamma\gamma - c\alpha \\ &= \sum_{\gamma\in\Delta\setminus\{\alpha\}} c_\gamma\gamma + (c_\alpha - c)\alpha. \end{aligned}$$

Since $s_\alpha\beta \in \Phi \subset \mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta$, (65) implies $s_\alpha\beta \in \Phi \cap \mathbf{R}_{\geq 0}\Delta = \pi(\Delta) = \Pi$. Since $\beta \in \Pi \not\ni -\alpha$, we have $\beta \neq -\alpha = s_\alpha\alpha$. Thus $s_\alpha\beta \neq \alpha$. Therefore, $s_\alpha\beta \in \Pi \setminus \{\alpha\}$. □

Definition 35. Let G be a group, and let Ω be a set. We say that G *acts on* Ω if there is a mapping

$$\begin{array}{ccc} G \times \Omega & \to & \Omega \\ (g, \alpha) & \mapsto & g.\alpha \end{array} \quad (g \in G,\ \alpha \in \Omega)$$

such that

(i) $1.\alpha = \alpha$ for all $\alpha \in \Omega$,

(ii) $g.(h.\alpha) = (gh).\alpha$ for all $g, h \in G$ and $\alpha \in \Omega$.

We say that G acts *transitively* on Ω, or the action of G is *transitive*, if

$$\forall \alpha, \beta \in \Omega,\ \exists g \in G,\ g.\alpha = \beta.$$

Observe, by Lemma 23,

$$|\Pi| = \frac{1}{2}|\Phi| \quad (\Pi \in \mathcal{P}(\Phi)). \tag{66}$$

Theorem 36. *The group $W(\Phi)$ acts transitively on both $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$.*

Proof. First we show that

$$\forall \Pi, \Pi' \in \mathcal{P}(\Phi),\ \exists w \in W(\Phi),\ w\Pi = \Pi' \tag{67}$$

by induction on $r = |\Pi \cap (-\Pi')|$. If $r = 0$, then $\Pi \subset \Pi'$, and we obtain $\Pi = \Pi'$ by (66).

If $r > 0$, then $\Pi \neq \Pi'$. Let $\Delta = \pi^{-1}(\Pi)$. Then $\Delta \neq \pi^{-1}(\Pi')$, so Δ is not contained in Π' by Theorem 30(ii). This implies $\Delta \cap (-\Pi') \neq \emptyset$ since $\Phi = \Pi' \cup (-\Pi')$. Choose $\alpha \in \Delta \cap (-\Pi')$. Then

$$-\alpha \notin -\Pi'. \tag{68}$$

Since

$$\begin{aligned} s_\alpha \Pi &= s_\alpha(\{\alpha\} \cup (\Pi \setminus \{\alpha\})) \\ &= \{s_\alpha \alpha\} \cup (s_\alpha(\Pi \setminus \{\alpha\})) \\ &= \{-\alpha\} \cup s_\alpha(\Pi \setminus \{\alpha\}) \\ &= \{-\alpha\} \cup (\Pi \setminus \{\alpha\}) && \text{(by Lemma 34),} \end{aligned}$$

we have

$$\begin{aligned} |s_\alpha \Pi \cap (-\Pi')| &= |(\{-\alpha\} \cup (\Pi \setminus \{\alpha\})) \cap (-\Pi')| \\ &= |(\Pi \setminus \{\alpha\}) \cap (-\Pi')| && \text{(by (68))} \\ &= |(\Pi \cap (-\Pi')) \setminus \{\alpha\}| \\ &= r - 1. \end{aligned}$$

Since $s_\alpha \Pi \in \mathcal{P}(\Phi)$ by Lemma 33(ii), the inductive hypothesis applied to the pair $s_\alpha \Pi, \Pi'$ implies that there exists $w \in W(\Phi)$ such that $w s_\alpha \Pi = \Pi'$. Therefore, we have proved (67), which implies that $W(\Phi)$ acts transitively on $\mathcal{P}(\Phi)$. The transitivity of $W(\Phi)$ on $\mathcal{S}(\Phi)$ now follows immediately from Lemma 33 using the fact that π is a bijection from $\mathcal{S}(\Phi)$ to $\mathcal{P}(\Phi)$. □

Definition 37. Let $\Delta \in \mathcal{S}(\Phi)$. For $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha \in \Phi$, the *height* of β relative to Δ, denoted $\mathrm{ht}(\beta)$, is defined as

$$\mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} c_\alpha.$$

Example 38. Continuing Example 26, let

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < n\} \in \mathcal{S}(\Phi),$$

where

$$\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\}.$$

Then for $i < j$,

$$\mathrm{ht}(\varepsilon_i - \varepsilon_j) = \mathrm{ht}(\sum_{k=i}^{j-1} (\varepsilon_k - \varepsilon_{k+1})) = j - i.$$