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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Recall that for $0 \neq \alpha \in V$, $s_\alpha \in O(V)$ denotes the reflection

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad (\lambda \in V).$$

Lemma 1. *For $t \in O(V)$ and $0 \neq \alpha \in V$, we have $ts_\alpha t^{-1} = s_{t\alpha}$.*

Definition 2. Let Φ be a nonempty finite set of nonzero vectors in V. We say that Φ is a *root system* if

(R1) $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,

(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Definition 3. A *total ordering* of V is a transitive relation on V (denoted $<$) satisfying the following axioms.

(i) For each pair $\lambda, \mu \in V$, exactly one of $\lambda < \mu$, $\lambda = \mu$, $\mu < \lambda$ holds.

(ii) For all $\lambda, \mu, \nu \in V$, $\mu < \nu$ implies $\lambda + \mu < \lambda + \nu$.

(iii) Let $\mu < \nu$ and $c \in \mathbf{R}$. If $c > 0$ then $c\mu < c\nu$, and if $c < 0$ then $c\nu < c\mu$.

For convenience, we write $\lambda > \mu$ if $\mu < \lambda$. By (ii), $\lambda > 0$ implies $0 > -\lambda$. Thus

$$V = V_+ \cup \{0\} \cup V_- \quad (\text{disjoint}),$$

where

$$V_+ = \{\lambda \in V \mid \lambda > 0\},$$
$$V_- = \{\lambda \in V \mid \lambda < 0\}.$$

Lemma 4. *Let $<$ be a total ordering of V, and let $\lambda, \mu \in V$.*

(i) *If $\lambda, \mu > 0$, then $\lambda + \mu > 0$.*

(ii) *If $\lambda > 0$, $c \in \mathbf{R}$ and $c > 0$, then $c\lambda > 0$.*

(iii) *If $\lambda > 0$, $c \in \mathbf{R}$ and $c < 0$, then $c\lambda < 0$. In particular, $-\lambda < 0$.*

Lemma 5. *Let Δ be a finite set of nonzero vectors in V_+. If $(\alpha, \beta) \leq 0$ for any distinct $\alpha, \beta \in \Delta$, then Δ consists of linearly independent vectors.*

Lemma 6. *Let $\Delta \subset V_+$ be a subset, and let $\alpha, \beta \in \Delta$ be linearly independent. If $\alpha \in \mathbf{R}_{>0}\beta + \mathbf{R}_{\geq 0}\Delta$, then $\alpha \in \mathbf{R}_{\geq 0}(\Delta \setminus \{\alpha\})$.*

Definition 7. Let Φ be a root system in V. A subset Π of Φ is called a *positive system* if there exists a total ordering $<$ of V such that $\Pi = \{\alpha \in \Phi \mid \alpha > 0\}$.

Lemma 8. *If Π is a positive system in a root system Φ, then $\Phi = \Pi \cup (-\Pi)$ (disjoint), where*

$$-\Pi = \{-\alpha \mid \alpha \in \Pi\}.$$

In particular,

$$-\Pi = \{\alpha \in \Phi \mid \alpha < 0\}.$$

Definition 9. Let Δ be a subset of a root system Φ. We call Δ a *simple system* if Δ is a basis of the subspace spanned by Φ, and if moreover $\Phi \subset \mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta$ holds.

In what follows, we fix a root system Φ in V. Recall that $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$ denote the set of positive systems and that of simple systems, respectively, in Φ.

Lemma 10. *If $\Delta \in \mathcal{S}(\Phi)$, $\Pi \in \mathcal{P}(\Phi)$ and $\Delta \subset \Pi$, then*

(i) $\Pi = \Phi \cap \mathbf{R}_{\geq 0}\Delta$,

(ii) $\Delta = \{\alpha \in \Pi \mid \alpha \notin \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})\}$.

Theorem 11. *The mapping*

$$\begin{array}{rcl} \pi : \mathcal{S}(\Phi) & \to & \mathcal{P}(\Phi) \\ \Delta & \mapsto & \Phi \cap \mathbf{R}_{\geq 0}\Delta \end{array}$$

is a bijection whose inverse is given by

$$\begin{array}{rcl} \pi^{-1} : \mathcal{P}(\Phi) & \to & \mathcal{S}(\Phi) \\ \Pi & \mapsto & \{\alpha \in \Pi \mid \alpha \notin \mathbf{R}_{\geq 0}(\Pi \setminus \{\alpha\})\}. \end{array} \tag{1}$$

Moreover,

(i) *for every simple system Δ in Φ, $\pi(\Delta)$ is the unique positive system containing Δ,*

(ii) *for every positive system Π in Φ, $\pi^{-1}(\Pi)$ is the unique simple system contained in Π.*

Example 12. Let $\varepsilon_1, \dots, \varepsilon_n$ be the standard basis of $\mathbf{R}^n$. The set

$$\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\}$$

is a root system, with a positive system

$$\Pi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\},$$

and simple system

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < n\}.$$