June 6, 2016

For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Recall that for $0 \neq \alpha \in V$, $s_\alpha \in O(V)$ denotes the reflection

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad (\lambda \in V).$$

Lemma 1. *For $t \in O(V)$ and $0 \neq \alpha \in V$, we have $ts_\alpha t^{-1} = s_{t\alpha}$.*

Definition 2. Let Φ be a root system in V. A subset Π of Φ is called a *positive system* if there exists a total ordering $<$ of V such that $\Pi = \{\alpha \in \Phi \mid \alpha > 0\}$.

Lemma 3. *If Π is a positive system in a root system Φ, then $\Phi = \Pi \cup (-\Pi)$ (disjoint).*

Definition 4. Let Δ be a subset of a root system Φ. We call Δ a *simple system* if Δ is a basis of the subspace spanned by Φ, and if moreover $\Phi \subset \mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta$ holds.

In what follows, we fix a root system Φ in V, a positive system Π and a simple system $\Delta \subset \Pi$.

Lemma 5. *For $\alpha \in \Delta$, $s_\alpha(\Pi \setminus \{\alpha\}) = \Pi \setminus \{\alpha\}$.*

Definition 6. For $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha \in \Phi$, the *height* of β relative to Δ, denoted $\operatorname{ht}(\beta)$, is defined as

$$\operatorname{ht}(\beta) = \sum_{\alpha \in \Delta} c_\alpha.$$

Definition 7. For $w \in W$, we define the *length* of w, denoted $\ell(w)$, to be

$$\ell(w) = \min\{r \in \mathbf{Z} \mid r \geq 0,\ \exists \alpha_1, \ldots, \alpha_r \in \Delta,\ w = s_{\alpha_1} \cdots s_{\alpha_r}\}.$$

By convention, $\ell(1) = 0$.

Notation 8. For $w \in W$, we write

$$n(w) = |\Pi \cap w^{-1}(-\Pi)|.$$

Definition 9. A linear transformation $s : \mathbf{R}^n \to \mathbf{R}^n$ is called a reflection if there exists a nonzero vector α such that $s(\alpha) = -\alpha$ and $s(h) = h$ for all $h \in (\mathbf{R}\alpha)^\perp$.

Lemma 10. *Let $s : \mathbf{R}^n \to \mathbf{R}^n$ be a reflection. Then the matrix representation S of s is diagonalizable by an orthogonal matrix:*

$$P^{-1}SP = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

for some orthogonal matrix P.

Example 11. Let $n \geq 2$ be an integer, and let $\mathcal{S}_n$ denote the symmetric group of degree n. In other words, $\mathcal{S}_n$ consists of all permutations of the set $\{1, 2, \dots, n\}$. Since permutations are bijections from $\{1, 2, \dots, n\}$ to itself, $\mathcal{S}_n$ forms a group under composition. Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of $\mathbf{R}^n$. For each $\sigma \in \mathcal{S}_n$, we define $g_\sigma \in O(\mathbf{R}^n)$ by setting

$$g_\sigma(\sum_{i=1}^{n} c_i\varepsilon_i) = \sum_{i=1}^{n} c_i\varepsilon_{\sigma(i)},$$

and set

$$G_n = \{g_\sigma \mid \sigma \in \mathcal{S}_n\}.$$

It is easy to verify that G_n is a subgroup of $O(V)$ and, the mapping $\mathcal{S}_n \to G_n$ defined by $\sigma \mapsto g_\sigma$ is an isomorphism. It is well known that $\mathcal{S}_n$ is generated by its set of transposition. Via the isomorphism $\sigma \mapsto g_\sigma$, we see that G_n is generated by the set of reflections

$$\{s_{\varepsilon_i - \varepsilon_j} \mid 1 \leq i < j \leq n\}. \tag{1}$$

The set

$$\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\}$$

is a root system, with a positive system

$$\Pi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}, \tag{2}$$

and simple system

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < n\}.$$

Exercise 12. Show that (1) is precisely the set of reflections in G_n. In other words, for $\sigma \in \mathcal{S}_n$, show that g_σ is a reflection if and only if σ is a transposition.

Exercise 13. With reference to Notation 8 and (2), show that

$$n(g_\sigma) = |\{(i, j) \mid i, j \in \{1, 2, \dots, n\},\ i < j,\ \sigma(i) > \sigma(j)\}| \quad (\sigma \in \mathcal{S}_n).$$

Exercises 12 and 13 are due on June 13, 2016.