Exercise 10. *Given a finite reflection group $W \subset O(V)$, let*

$$U = \{\lambda \in V \mid \forall t \in W,\ t\lambda = \lambda\}.$$

Let U' denote the orthogonal complement of U in V. Then show that the restriction homomorphism $W \to O(U')$ defined by $t \mapsto t|_{U'}$ is injective, and the image $W|_{U'}$ is an essential finite reflection group in $O(U')$.

Proof. For notational convenience, let $\varphi : W \to O(U')$ denote the restriction homomorphism, that is, $\varphi(t) = t|_{U'}$ for $t \in W$.

First we show that φ is injective. Suppose $s, t \in W$ and $\varphi(s) = \varphi(t)$. Given $\lambda \in V$, there exist vectors $\lambda_1 \in U$ and $\lambda_2 \in U'$ such that $\lambda = \lambda_1 + \lambda_2$ since U' is the orthogonal complement of U in V. Then

$$\begin{aligned}
s\lambda &= s(\lambda_1 + \lambda_2) \\
&= s\lambda_1 + s\lambda_2 \\
&= \lambda_1 + s\lambda_2 && \text{(by } \lambda_1 \in U\text{)} \\
&= \lambda_1 + t\lambda_2 && \text{(by } \lambda_2 \in U' \text{ and } \varphi(s) = \varphi(t)\text{)} \\
&= t\lambda_1 + t\lambda_2 && \text{(by } \lambda_1 \in U\text{)} \\
&= t(\lambda_1 + \lambda_2) \\
&= t\lambda.
\end{aligned}$$

Therefore $s = t$, so that the restriction homomorphism is injective.

Next we show that the image $W|_{U'}$ is a finite reflection group in $O(U')$. It is clearly a subgroup of $O(U')$ by its construction. Since W is a finite reflection group W,

(i) $W \neq \{\mathrm{id}_V\}$,

(ii) W is finite,

(iii) W is generated by a set of reflections.

Since the restriction homomorphism φ is injective, (i) implies $W|_{U'} \neq \{\mathrm{id}_V\}$, while $W|_{U'}$ is finite by (ii). To see that $W|_{U'}$ is generated by a set of reflections, because of (iii), it suffices show that $\varphi(s)$ is a reflection for whenever $s \in W$ is a reflection. If $s \in W$ is a reflection, then there exists a nonzero vector $\alpha \in V$ such that $s\alpha = -\alpha$ and $sh = h$ for all $h \in (\mathbf{R}\alpha)^\perp$. This implies $U \subset (\mathbf{R}\alpha)^\perp$, and hence $\alpha \in U'$. In particular, $\varphi(s)$ is a reflection in U'. We have now proved that the image $W|_{U'}$ is a finite reflection group in $O(U')$.

Finally we show that the image $W|_{U'}$ is essential. Suppose that $\lambda \in U'$ satisfies $t'\lambda = \lambda$ for all $t' \in W|_{U'}$. Then $t\lambda = \lambda$ for all $t \in W$, which implies $\lambda \in U$. Therefore, $\lambda \in U \cap U' = \{0\}$. This proves that the image $W|_{U'}$ is essential. $\square$

Exercise 11. *Let $\mathcal{S}_3$ denote the symmetric group of order 3 and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ denote the standard basis of $\mathbf{R}^3$. For each $\sigma \in \mathcal{S}_3$, we define $g_\sigma \in O(\mathbf{R}^3)$ by $g_\sigma(\sum_{i=1}^3 c_i\varepsilon_i) = \sum_{i=1}^3 c_i\varepsilon_{\sigma(i)}$, and set $G_3 = \{g_\sigma \mid \sigma \in \mathcal{S}_3\}$. Moreover we set $\eta_1 = \frac{1}{\sqrt{2}}(\varepsilon_1 - \varepsilon_2)$ and $\eta_2 = \frac{1}{\sqrt{6}}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3)$. Compute the matrix representations of $g_{(1\ 2)}$ and $g_{(2\ 3)}$ with respect to the basis $\{\eta_1, \eta_2\}$. Show that they are reflections whose lines of symmetry form an angle $\pi/3$.*

Proof. By definition,

$$
\begin{aligned}
g_{(1\ 2)}(\eta_1) &= g_{(1\ 2)}\left(\frac{1}{\sqrt{2}}(\varepsilon_1 - \varepsilon_2)\right) = \frac{1}{\sqrt{2}}(\varepsilon_2 - \varepsilon_1) = -\eta_1,\\
g_{(1\ 2)}(\eta_2) &= g_{(1\ 2)}\left(\frac{1}{\sqrt{6}}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3)\right) = \frac{1}{\sqrt{6}}(\varepsilon_2 + \varepsilon_1 - 2\varepsilon_3) = \eta_2,\\
g_{(2\ 3)}(\eta_1) &= g_{(2\ 3)}\left(\frac{1}{\sqrt{2}}(\varepsilon_1 - \varepsilon_2)\right) = \frac{1}{\sqrt{2}}(\varepsilon_1 - \varepsilon_3) = \frac{1}{2}\eta_1 + \frac{\sqrt{3}}{2}\eta_2,\\
g_{(2\ 3)}(\eta_2) &= g_{(2\ 3)}\left(\frac{1}{\sqrt{6}}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3)\right) = \frac{1}{\sqrt{6}}(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2) = \frac{\sqrt{3}}{2}\eta_1 - \frac{1}{2}\eta_2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\begin{pmatrix} g_{(1\ 2)}(\eta_1) & g_{(1\ 2)}(\eta_2) \end{pmatrix} &= \begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\\
\begin{pmatrix} g_{(2\ 3)}(\eta_1) & g_{(2\ 3)}(\eta_2) \end{pmatrix} &= \begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.
\end{aligned}
$$

Hence the matrix representations of $g_{(1\ 2)}$ and $g_{(2\ 3)}$ with respect to the basis $\{\eta_1, \eta_2\}$ is given by

$$
\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},
$$

respectively.

It is easy to see that $g_{(1\ 2)}$ is a reflection with respect to the y-axis which forms an angle $\pi/2$ with the x-axis. Indeed,

$$
\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\pi & \sin\pi \\ \sin\pi & -\cos\pi \end{pmatrix}.
$$

Similarly, $g_{(2\ 3)}$ is a reflection with respect to a line L which forms an angle $\pi/6$ with the x-axis, since

$$
\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & -\cos\frac{\pi}{3} \end{pmatrix}.
$$

Moreover, the y axis and the line L form an angle

$$
\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.
$$

□