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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Recall that for $0 \neq \alpha \in V$, $s_\alpha \in O(V)$ denotes the reflection

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad (\lambda \in V).$$

Definition 1. Let Φ be a nonempty finite set of nonzero vectors in V. We say that Φ is a *root system* if

(R1) $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,

(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Let Φ be a root system in V, and let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$.

Definition 2. Let Φ be a root system in V. A subset Π of Φ is called a *positive system* if there exists a total ordering $<$ of V such that $\Pi = \{\alpha \in \Phi \mid \alpha > 0\}$.

Definition 3. Let Δ be a subset of a root system Φ. We call Δ a *simple system* if Δ is a basis of the subspace spanned by Φ, and if moreover $\Phi \subset \mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta$ holds.

Theorem 4. *If Δ is a simple system in a root system Φ, then $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$.*

Recall that $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$ denote the set of positive systems and that of simple systems, respectively, in Φ.

Lemma 5. *Let $w \in W$. Then*

(i) $w\Delta \in \mathcal{S}(\Phi)$ *and* $\pi(w\Delta) = w\pi(\Delta)$ *for all* $\Delta \in \mathcal{S}(\Phi)$,

(ii) $w\Pi \in \mathcal{P}(\Phi)$ *and* $\pi^{-1}(w\Pi) = w\pi^{-1}(\Pi)$ *for all* $\Pi \in \mathcal{P}(\Phi)$.

Theorem 6. *The group W acts transitively on both $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$.*

Notation 7. For $w \in W$, we write

$$n(w) = |\Pi \cap w^{-1}(-\Pi)|.$$

Definition 8. For $w \in W$, we define the *length* of w, denoted $\ell(w)$, to be

$$\ell(w) = \min\{r \in \mathbf{Z} \mid r \geq 0,\ \exists \alpha_1, \dots, \alpha_r \in \Delta,\ w = s_{\alpha_1} \cdots s_{\alpha_r}\}.$$

By convention, $\ell(1) = 0$.

Corollary 9. *If $w \in W$, then $n(w) = \ell(w)$.*