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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Let Φ be a root system in V with simple system Δ. Let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$. Recall Notation 56.

Lemma 59. *Let $I \subset S$. If $u \in W$ satisfies*

$$\ell(u) = \min\{\ell(x) \mid x \in uW_I\},$$

then

$$\ell(uv) = \ell(u) + \ell(v) \quad (\forall v \in W_I).$$

Proof. Let $q = \ell(u)$. Then there exist $s_1, \ldots, s_q \in S$ such that

$$u = s_1 \cdots s_q.$$

Let $v \in W_I$. Then by Proposition 58(iv), we have $\ell(v) = \ell_I(v)$. This implies that there exist $s_{q+1}, \ldots, s_{q+r} \in I$ such that

$$v = s_{q+1} \cdots s_{q+r},$$

where $r = \ell(v)$. Then $uv = s_1 \cdots s_{q+r}$, hence $\ell(uv) \le q + r$.

Suppose $\ell(w) < q + r$. Then by Theorem 48, there exist i, j with $1 \le i < j \le q + r$ such that

$$uv = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_{q+r}.$$

If $i < j \le q$, then

$$uv = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_q v,$$

hence $u = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_q$, contradicting $\ell(u) = q$. Similarly, if $q + 1 \le i < j$, then

$$uv = us_{q+1} \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_{q+r},$$

hence $v = s_{q+1} \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_{q+r}$, contradicting $\ell(v) = r$. Thus

$$1 \le i \le q < j \le q + r.$$

Setting

$$u' = s_1 \cdots \hat{s}_i \cdots s_q,$$
$$v' = s_{q+1} \cdots \hat{s}_j \cdots s_{q+r} \in W_I,$$

we have $u'v' = uv$, and hence $u' = uvv'^{-1} \in uW_I$. But $\ell(u') < q = \ell(u)$, contrary to the minimality of $\ell(u)$. Therefore, we conclude $\ell(w) = q + r = \ell(u) + \ell(v)$. $\square$

Notation 60. For $I \subset S$, we define

$$W^I = \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}.$$

Lemma 61. *Let $I \subset S$ and $w \in W$. If $u_0 \in wW_I$ satisfies*

$$\ell(u_0) = \min\{\ell(x) \mid x \in wW_I\},$$

and $u_1 \in W^I \cap wW_I$, then $u_0 = u_1$. In particular,

(i) *$W^I \cap wW_I$ consists of a single element,*

(ii) *$\min\{\ell(x) \mid x \in wW_I\}$ is achieved by a unique element,*

and the elements described in (i) *and* (ii) *coincide.*

Proof. Since $u_1 \in wW_I = u_0W_I$, there exists $v \in W_I$ such that $u_1 = u_0v$. Suppose $v \neq 1$. Then there exists $s \in I$ such that $\ell(vs) < \ell(v)$. This implies

$$\begin{aligned} \ell(u_1s) &= \ell(u_0vs) \\ &= \ell(u_0) + \ell(vs) && \text{(by Lemma 59)} \\ &< \ell(u_0) + \ell(v) \\ &= \ell(u_0v) && \text{(by Lemma 59)} \\ &= \ell(u_1). \end{aligned}$$

This contradicts $u_1 \in W^I$. Thus, we conclude $v = 1$, or equivalently, $u_1 = u_0$. The rest of the statements are immediate. □

Lemma 62. *Let $I \subset S$. The mapping $\phi : W^I \times W_I \to W$ defined by $\phi(u, v) = uv$ is a bijection, and it satisfies*

$$\ell(\phi(u, v)) = \ell(u) + \ell(v) \quad (u \in W^I,\ v \in W_I).$$

Proof. Let $w \in W$. Choose $u_0 = u_1 \in W^I \cap wW_I$ as in Lemma 61. Then there exists $v \in W_I$ such that $u_0 = wv$. Then $w = \phi(u_0, v^{-1})$. Thus ϕ is surjective.

Suppose $(u, v), (u', v') \in W^I \times W_I$ and $\phi(u, v) = \phi(u', v')$. Then $uv = u'v'$. Thus $u, u' \in W^I \cap uW_I$, which forces $u = u'$ by Lemma 61(i). Then we also have $v = v'$. Thus ϕ is injective.

Finally, for $u \in W^I$, we have $u \in W^I \cap uW_I$, so Lemma 61 implies $\ell(u) = \min\{\ell(x) \mid x \in uW_I\}$. Then by Lemma 59, we have $\ell(uv) = \ell(u) + \ell(v)$ for all $v \in W_I$. □

Notation 63. Let t be an indeterminate over $\mathbf{Q}$, or in other words, consider the polynomial ring $\mathbf{Q}[t]$ (or its field of fractions $\mathbf{Q}(t)$). For a subset X of W, write

$$X(t) = \sum_{w \in X} t^{\ell(w)}.$$

Definition 64. The *Poincaré polynomial* $W(t)$ of W is defined as

$$W(t) = \sum_{w \in W} t^{\ell(w)}.$$

We remark that $W(t)$ is independent of the choice of a simple system, even though the length function ℓ does depend on it. Indeed, let Δ' be another simple system. Then there exists $z \in W$ such that $\Delta' = z\Delta$ by Theorem 36. Let

$$S = \{s_\alpha \mid \alpha \in \Delta\},$$
$$S' = \{s_\alpha \mid \alpha \in \Delta'\}.$$

Then

$$\begin{aligned} zSz^{-1} &= \{zs_\alpha z^{-1} \mid \alpha \in \Delta\} \\ &= \{s_{z\alpha} \mid \alpha \in \Delta\} && \text{(by Lemma 12)} \\ &= \{s_\alpha \mid \alpha \in z\Delta\} \\ &= \{s_\alpha \mid \alpha \in \Delta'\} \\ &= S'. \end{aligned}$$

If we denote by the length function with respect to Δ and Δ' by ℓ_Δ and $\ell_{\Delta'}$, respectively, then $\ell_\Delta(w) = \ell_{\Delta'}(zwz^{-1})$ for all $w \in W$. Thus

$$\sum_{w\in W} t^{\ell_\Delta(w)} = \sum_{w\in W} t^{\ell_{\Delta'}(zwz^{-1})} = \sum_{w\in W} t^{\ell_{\Delta'}(w)}.$$

Lemma 65. *For $I \subset S$,*

$$W(t) = W^I(t)W_I(t).$$

Proof. By Lemma 62,

$$\begin{aligned} W(t) &= \sum_{w\in W} t^{\ell(w)} \\ &= \sum_{(u,v)\in W^I\times W_I} t^{\ell(\phi(u,v))} \\ &= \sum_{u\in W^I}\sum_{v\in W_I} t^{\ell(u)+\ell(v)} \\ &= \sum_{u\in W^I} t^{\ell(u)} \sum_{v\in W_I} t^{\ell(v)} \\ &= W^I(t)W_I(t). \end{aligned}$$

□

Lemma 66. *Let Π be the unique positive system containing Δ. For $w \in W$, set*

$$K(w) = \{s \in S \mid \ell(ws) > \ell(w)\}.$$

Then the following are equivalent:

(i) $K(w) = \emptyset$,

(ii) $w\Pi = -\Pi$,

(iii) $\ell(w) = |\Pi|$.

Moreover, there exists a unique $w \in W$ satisfying these conditions.

Proof. Equivalence of (ii) and (iii) follows from Corollary 49.

$$\begin{aligned}
\text{(i)} &\iff \ell(ws) < \ell(w) \quad (\forall s \in S) \\
&\iff w\Delta \subset -\Pi && \text{(by Lemma 47)} \\
&\iff w\Pi \subset -\Pi \\
&\iff \text{(ii)}.
\end{aligned}$$

The uniqueness of w follows from Theorem 55. □

Proposition 67. *Then*

$$\sum_{I \subset S} (-1)^{|I|} \frac{W(t)}{W_I(t)} = \sum_{I \subset S} (-1)^{|I|} W^I(t) = t^{|\Pi|}.$$

Proof. The first equality follows immediately from Lemma 65. For $I \subset S$, we have

$$w \in W^I \iff K(w) \supset I.$$

Thus

$$\begin{aligned}
\sum_{I \subset S} (-1)^{|I|} W^I(t) &= \sum_{I \subset S} (-1)^{|I|} \sum_{w \in W^I} t^{\ell(w)} \\
&= \sum_{w \in W} \sum_{\substack{I \subset S \\ w \in W^I}} (-1)^{|I|} t^{\ell(w)} \\
&= \sum_{w \in W} \sum_{I \subset K(w)} (-1)^{|I|} t^{\ell(w)} \\
&= \sum_{w \in W} t^{\ell(w)} \sum_{i=0}^{|K(w)|} \sum_{\substack{I \subset K(w) \\ |I| = i}} (-1)^i \\
&= \sum_{w \in W} t^{\ell(w)} \sum_{i=0}^{|K(w)|} (-1)^i \binom{|K(w)|}{i} \\
&= \sum_{\substack{w \in W \\ |K(w)| = 0}} t^{\ell(w)} + \sum_{\substack{w \in W \\ |K(w)| \geq 1}} t^{\ell(w)} (1 + (-1))^{|K(w)|} \\
&= \sum_{\substack{w \in W \\ K(w) = \emptyset}} t^{\ell(w)} \\
&= t^{|\Pi|}
\end{aligned}$$

by Lemma 66. □