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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Let Φ be a root system in V with simple system Δ. and let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$. Let $\Pi = \Phi \cap \mathbf{R}_{\geq 0}\Delta$ be the unique positive system in Φ containing Δ.

Lemma 1. *For $t \in O(V)$ and $0 \neq \alpha \in V$, we have $ts_\alpha t^{-1} = s_{t\alpha}$.*

Theorem 2. $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$.

Definition 3. For $w \in W$, we define the *length* of w, denoted $\ell(w)$, to be

$$\ell(w) = \min\{r \in \mathbf{Z} \mid r \geq 0,\ \exists \alpha_1, \ldots, \alpha_r \in \Delta,\ w = s_{\alpha_1} \cdots s_{\alpha_r}\}.$$

By convention, $\ell(1) = 0$.

Lemma 4. *For $w \in W$ and $\alpha \in \Delta$, the following statements hold:*

(i) $w\alpha > 0 \implies \ell(ws_\alpha) = \ell(w) + 1$.

(ii) $w\alpha < 0 \implies \ell(ws_\alpha) = \ell(w) - 1$.

Theorem 5. *Let $\alpha_1, \ldots, \alpha_r \in \Delta$ and $w = s_1 \cdots s_r \in W$, where $s_i = s_{\alpha_i}$ for $1 \leq i \leq r$. If $\ell(w) < r$, then there exist i, j with $1 \leq i < j \leq r$ such that*

$$w = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_r.$$

Notation 6. For $w \in W$, we write

$$n(w) = |\Pi \cap w^{-1}(-\Pi)|.$$

Corollary 7. *If $w \in W$, then $n(w) = \ell(w)$.*

Theorem 8. *The group $W(\Phi)$ acts simply transitively on $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$.*

Notation 9. Let $S = \{s_\alpha \mid \alpha \in \Delta\}$. For $I \subset S$, we define

$$\begin{aligned} W_I &= \langle I \rangle, \\ \Delta_I &= \{\alpha \in \Delta \mid s_\alpha \in I\}, \\ V_I &= \mathbf{R}\Delta_I, \\ \Phi_I &= \Phi \cap V_I, \\ \Pi_I &= \Pi \cap V_I. \end{aligned}$$

Proposition 10. *Let $I \subset S$.*

(i) Φ_I *is a root system with simple system* Δ_I*.*

(ii) Π_I *is the unique positive system of* Φ_I *containing the simple system* Δ_I*.*

(iii) $W(\Phi_I) = W_I$.

(iv) *Let ℓ be the length function of W with respect to Δ. Then the restriction of ℓ to W_I coincides with the length function ℓ_I of W_I with respect to the simple system Δ_I.*