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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Let Φ be a root system in V with simple system Δ, and let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi\rangle$. Let $\Pi = \Phi \cap \mathbf{R}_{\geq 0}\Delta$ be the unique positive system in Φ containing Δ.

Recall Notation 56 and Proposition 67:

$$\sum_{I \subsetneqq S} \frac{(-1)^{|I|}}{W_I(t)} = \frac{t^{|\Pi|} - (-1)^{|S|}}{W(t)}. \tag{92}$$

Continuing Example 16 with $n = 4$, write $W = G_4$, $s_i = s_{\varepsilon_i - \varepsilon_{i+1}}$ for $i = 1, 2, 3$, so that $S = \{s_1, s_2, s_3\}$. Then

$$\begin{aligned} W_\emptyset(t) &= 1, \\ W_{\{s_i\}}(t) &= t + 1, \\ W_{\{s_1, s_2\}}(t) &= (t+1)(t^2 + t + 1). \end{aligned}$$

If we compute $W_I(t)$ for all $I \subsetneqq S$, then (92) can be used to determine $W(t)$ and, in particular, $|W|$.

Define

$$\begin{aligned} C &= \{\lambda \in V \mid (\lambda, \alpha) > 0 \ (\forall \alpha \in \Delta)\}, \\ D &= \{\lambda \in V \mid (\lambda, \alpha) \geq 0 \ (\forall \alpha \in \Delta)\}. \end{aligned}$$

Lemma 68. *For each $\lambda \in V$, there exist $w \in W$ such that $w\lambda \in D$. Moreover, in this case, $w\lambda - \lambda \in \mathbf{R}_{\geq 0}\Delta$.*

Proof. Let $\lambda \in V$. Define a partial order on the set $W\lambda = \{w\lambda \mid w \in W\}$ by setting

$$\mu \preceq \mu' \iff \mu' - \mu \in \mathbf{R}_{\geq 0}\Delta \quad (\mu, \mu' \in W\lambda).$$

Since $W\lambda$ is finite, so is its subset

$$M = \{\mu \in W\lambda \mid \mu \geq \lambda\}.$$

The set M is non-empty since $\lambda \in M$. Thus, there exists a maximal element μ in M. Since $\mu = w\lambda$ for some $w \in W$ and $\mu - \lambda \in \mathbf{R}_{\geq 0}\Delta$, it remains to show $\mu \in D$.

Suppose, to the contrary, $\mu \notin D$. Then there exists $\alpha \in \Delta$ such that $(\mu, \alpha) < 0$. By the definition of a reflection, we have $s_\alpha\mu - \mu \in \mathbf{R}_{>0}\alpha$, so

$$\begin{aligned} s_\alpha\mu - \lambda &= (s_\alpha\mu - \mu) + (\mu - \lambda) \\ &\in \mathbf{R}_{>0}\alpha + \mathbf{R}_{\geq 0}\Delta \\ &\subset \mathbf{R}_{\geq 0}\Delta \setminus \{0\}. \end{aligned}$$

This implies $s_\alpha\mu \geq \lambda$ and $s_\alpha\mu \neq \lambda$. Moreover, $s_\alpha\mu = s_\alpha w\lambda \in W\lambda$. Therefore, $s_\alpha\mu \in M$, and this contradicts maximality of μ in M. $\square$

Notation 69. For a subset U of V, define

$$\mathrm{Stab}_W(U) = \{w \in W \mid w\lambda = \lambda \ (\forall \lambda \in U)\}.$$

Lemma 70. (i) *If* $\lambda \in D$*, then*

$$\mathrm{Stab}_W(\{\lambda\}) = \langle s_\alpha \mid \alpha \in \Delta,\ s_\alpha\lambda = \lambda \rangle.$$

(ii) *If* $\lambda, \mu \in D$*,* $w \in W$ *and* $w\lambda = \mu$*, then* $\lambda = \mu$*.*

(iii) *If* $\lambda \in C$*, then* $\mathrm{Stab}_W(\{\lambda\}) = \{1\}$*.*

(iv) *If* $\lambda \in V$*, then*

$$\mathrm{Stab}_W(\{\lambda\}) = \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha\lambda = \lambda \rangle.$$

Proof. First we prove, for $w \in W$,

$$\lambda, \mu \in D,\ w\lambda = \mu \implies \lambda = \mu,\ w \in \langle s_\alpha \mid \alpha \in \Delta,\ s_\alpha\lambda = \lambda \rangle, \tag{93}$$

$$\lambda \in C,\ \mu \in D,\ w\lambda = \mu \implies w = 1 \tag{94}$$

by induction on $n(w) = |w\Pi \cap (-\Pi)|$. If $n(w) = 0$, then $\ell(w) = 0$ by Corollary 49, hence $w = 1$. Then (93) holds. Suppose $n(w) > 0$. Then there exists $\beta \in \Pi$ such that $w\beta \in -\Pi$. Since $\Pi \subset \mathbf{R}_{\geq 0}\Delta$, this implies $w\mathbf{R}_{\geq 0}\Delta \cap \mathbf{R}_{\leq 0}\Delta \neq \emptyset$, which in turn implies $w\Delta \cap (-\Pi) \neq \emptyset$. Suppose $w\gamma \in -\Pi$, where $\gamma \in \Delta$. Then by Lemma 47,

$$\begin{aligned} \ell(ws_\gamma) &= \ell(w) - 1 \\ &= n(w) - 1 && \text{(by Corollary 49)} \\ &< n(w). \end{aligned} \tag{95}$$

Since $\mu \in D$ and $-w\gamma \in \Pi \subset \mathbf{R}_{\geq 0}\Delta$, we have

$$\begin{aligned} 0 &\leq (\mu, -w\gamma) \\ &= -(w^{-1}\mu, \gamma) \\ &= -(\lambda, \gamma). \end{aligned}$$

If $\lambda \in C$, this is impossible. This implies that (94) holds. If $\lambda \in D$, then this forces $(\lambda, \gamma) = 0$, implying $s_\gamma \in \mathrm{Stab}_W(\{\lambda\})$. Now, we have $ws_\gamma\lambda = \mu$ and (95), so we can apply inductive hypothesis to conclude $\lambda = \mu$ and

$$ws_\gamma \in \langle s_\alpha \mid \alpha \in \Delta,\ s_\alpha\lambda = \lambda \rangle.$$

Thus (93) holds.

Now (ii) follows from (93), while (i) and (iii) follow from (93) and (94), respectively, by setting $\lambda = \mu$.

Finally we prove (iv). Let $\lambda \in V$. Clearly,

$$\mathrm{Stab}_W(\{\lambda\}) \supset \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha\lambda = \lambda \rangle.$$

To prove the reverse containment, observe that, by Lemma 68, there exists $z \in W$ such that $z\lambda \in D$. Then

$$\begin{aligned}
\operatorname{Stab}_W(\{\lambda\}) &= \{w \in W \mid w\lambda = \lambda\} \\
&= \{w \in W \mid zwz^{-1}z\lambda = z\lambda\} \\
&= \{z^{-1}xz \in W \mid xz\lambda = z\lambda\} \\
&= z^{-1} \operatorname{Stab}_W(\{z\lambda\}) z \\
&= z^{-1}\langle s_\beta \mid \beta \in \Delta,\ s_\beta z\lambda = z\lambda\rangle z && \text{(by (i))} \\
&= \langle z^{-1}s_\beta z \mid \beta \in \Delta,\ z^{-1}s_\beta z\lambda = \lambda\rangle \\
&= \langle s_{z^{-1}\beta} \mid \beta \in \Delta,\ s_{z^{-1}\beta}\lambda = \lambda\rangle && \text{(by Lemma 12)} \\
&\subset \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha\lambda = \lambda\rangle.
\end{aligned}$$

□

The following property of the set D is referred to as D being a *fundamental domain* for the action of W on V.

Theorem 71. *For each $\lambda \in V$, $|W\lambda \cap D| = 1$.*

Proof. By Lemma 68, we have $W\lambda \cap D \neq \emptyset$. Suppose $\mu, \mu' \in W\lambda \cap D$. Then Lemma 70(ii) implies $\mu = \mu'$. □