July 4, 2016

For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Let Φ be a root system in V with simple system Δ, and let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$. Let $\Pi = \Phi \cap \mathbf{R}_{\geq 0}\Delta$ be the unique positive system in Φ containing Δ.

Lemma 1. *For $t \in O(V)$ and $0 \neq \alpha \in V$, we have $ts_\alpha t^{-1} = s_{t\alpha}$.*

Theorem 2. $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$.

Definition 3. For $w \in W$, we define the *length* of w, denoted $\ell(w)$, to be

$$\ell(w) = \min\{r \in \mathbf{Z} \mid r \geq 0,\ \exists \alpha_1, \ldots, \alpha_r \in \Delta,\ w = s_{\alpha_1} \cdots s_{\alpha_r}\}.$$

By convention, $\ell(1) = 0$.

Lemma 4. *For $w \in W$ and $\alpha \in \Delta$, the following statements hold:*

(i) $w\alpha > 0 \implies \ell(ws_\alpha) = \ell(w) + 1$.

(ii) $w\alpha < 0 \implies \ell(ws_\alpha) = \ell(w) - 1$.

Notation 5. For $w \in W$, we write

$$n(w) = |\Pi \cap w^{-1}(-\Pi)|.$$

Corollary 6. *If $w \in W$, then $n(w) = \ell(w)$.*

Notation 7. Let $S = \{s_\alpha \mid \alpha \in \Delta\}$. For $I \subset S$, we define

$$\begin{aligned} W_I &= \langle I \rangle, \\ \Delta_I &= \{\alpha \in \Delta \mid s_\alpha \in I\}, \\ V_I &= \mathbf{R}\Delta_I, \\ \Phi_I &= \Phi \cap V_I, \\ \Pi_I &= \Pi \cap V_I. \end{aligned}$$

Proposition 8. *Let $I \subset S$.*

(i) *Φ_I is a root system with simple system Δ_I.*

(ii) *Π_I is the unique positive system of Φ_I containing the simple system Δ_I.*

(iii) $W(\Phi_I) = W_I$.

(iv) *Let ℓ be the length function of W with respect to Δ. Then the restriction of ℓ to W_I coincides with the length function ℓ_I of W_I with respect to the simple system Δ_I.*

Notation 9. Let t be an indeterminate over $\mathbf{Q}$, or in other words, consider the polynomial ring $\mathbf{Q}[t]$ (or its field of fractions $\mathbf{Q}(t)$). For a subset X of W, write

$$X(t) = \sum_{w \in X} t^{\ell(w)}.$$

Proposition 10. *Then*

$$\sum_{I \subset S} (-1)^{|I|} \frac{W(t)}{W_I(t)} = t^{|\Pi|}.$$

Example 11. Let $n \geq 2$ be an integer, and let $\mathcal{S}_n$ denote the symmetric group of degree n. In other words, $\mathcal{S}_n$ consists of all permutations of the set $\{1, 2, \ldots, n\}$. Since permutations are bijections from $\{1, 2, \ldots, n\}$ to itself, $\mathcal{S}_n$ forms a group under composition. Let $\varepsilon_1, \ldots, \varepsilon_n$ denote the standard basis of $\mathbf{R}^n$. For each $\sigma \in \mathcal{S}_n$, we define $g_\sigma \in O(\mathbf{R}^n)$ by setting

$$g_\sigma(\sum_{i=1}^{n} c_i \varepsilon_i) = \sum_{i=1}^{n} c_i \varepsilon_{\sigma(i)},$$

and set

$$G_n = \{g_\sigma \mid \sigma \in \mathcal{S}_n\}.$$

It is easy to verify that G_n is a subgroup of $O(V)$ and, the mapping $\mathcal{S}_n \to G_n$ defined by $\sigma \mapsto g_\sigma$ is an isomorphism. It is well known that $\mathcal{S}_n$ is generated by its set of transposition. Via the isomorphism $\sigma \mapsto g_\sigma$, we see that G_n is generated by the set of reflections

$$\{s_{\varepsilon_i - \varepsilon_j} \mid 1 \leq i < j \leq n\}. \tag{1}$$

The set

$$\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\}$$

is a root system, with a positive system

$$\Pi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}, \tag{2}$$

and simple system

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < n\}.$$

Exercise 12. Set $n = 4$ in Example 11 and compute the polynomial $W(t)$ using Proposition 10.