July 11, 2016

For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Let Φ be a root system in V with simple system Δ, and let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$.

Notation 72. Let $\alpha \in \Phi$. We define

$$\begin{aligned} H_\alpha &= \{\lambda \in V \mid (\alpha, \lambda) = 0\}, \\ H_\alpha^+ &= \{\lambda \in V \mid (\alpha, \lambda) > 0\}, \\ H_\alpha^- &= \{\lambda \in V \mid (\alpha, \lambda) < 0\}, \end{aligned}$$

so that $V = H_\alpha^+ \cup H_\alpha \cup H_\alpha^-$ (disjoint).

Recall

$$\begin{aligned} C &= \bigcap_{\alpha \in \Delta} H_\alpha^+, \\ D &= \bigcap_{\alpha \in \Delta} (H_\alpha^+ \cup H_\alpha). \end{aligned}$$

Lemma 73. *For $w \in W$ and $\alpha \in \Phi$,*

$$wH_\alpha = H_{w\alpha}, \tag{96}$$

$$wH_\alpha^\pm = H_{w\alpha}^\pm. \tag{97}$$

In particular,

$$s_\alpha H_\alpha^\pm = H_\alpha^\mp, \tag{98}$$

$$\bigcup_{\alpha \in \Phi} H_\alpha = w \bigcup_{\alpha \in \Phi} H_\alpha. \tag{99}$$

Proof. Observe

$$\begin{aligned} wH_\alpha &= \{w\lambda \mid \lambda \in V,\ (\alpha, \lambda) = 0\} \\ &= \{\mu \mid \mu \in V,\ (w\alpha, \mu) = 0\} \\ &= H_{w\alpha}. \end{aligned}$$

This proves (96). Replacing "=" by ">" or "<", we obtain (97). Moreover, (97) implies

$$\begin{aligned} s_\alpha H_\alpha^\pm &= H_{s_\alpha \alpha}^\pm \\ &= H_{-\alpha}^\pm \\ &= H_\alpha^\mp, \end{aligned}$$

while (96) implies

$$w \bigcup_{\alpha \in \Phi} H_\alpha = \bigcup_{\alpha \in \Phi} wH_\alpha$$

$$
\begin{aligned}
&= \bigcup_{\alpha\in\Phi} H_{w\alpha} \\
&= \bigcup_{\alpha\in w\Phi} H_{\alpha} \\
&= \bigcup_{\alpha\in\Phi} H_{\alpha}.
\end{aligned}
$$

□

Lemma 74. *If U is a linear subspace of V such that $\Phi \cap U \neq \emptyset$, then $\Phi \cap U$ is a root system.*

Proof. Clearly, $\Phi \cap U$ satisfies (R1) of Definition 14. As for (R2), let $\alpha, \beta \in \Phi \cap U$. Then $s_\alpha\beta \in \Phi \cap (\mathbf{R}\alpha + \mathbf{R}\beta) \subset \Phi \cap U$. Thus $s_\alpha(\Phi \cap U) \subset \Phi \cap U$. This implies $s_\alpha(\Phi \cap U) = \Phi \cap U$. □

Lemma 75. *If U is a linear subspace of V, then*

$$
\mathrm{Stab}_W(U) = \begin{cases} W(\Phi \cap U^\perp) & \text{if } \Phi \cap U^\perp \neq \emptyset, \\ \{1\} & \text{otherwise.} \end{cases}
$$

Proof. We prove the assertion by induction on $\dim U$. The assertion is trivial if $\dim U = 0$. If $\dim U = 1$, then write $U = \mathbf{R}\lambda$. We have

$$
\begin{aligned}
\mathrm{Stab}_W(U) &= \mathrm{Stab}_W(\{\lambda\}) \\
&= \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha\lambda = \lambda \rangle \qquad \text{(by Lemma 70(iv))} \\
&= \langle s_\alpha \mid \alpha \in \Phi,\ (\alpha, \lambda) = 0 \rangle \\
&= \langle s_\alpha \mid \alpha \in \Phi \cap (\mathbf{R}\lambda)^\perp \rangle \\
&= \begin{cases} W(\Phi \cap U^\perp) & \text{if } \Phi \cap U^\perp \neq \emptyset, \\ \{1\} & \text{otherwise,} \end{cases}
\end{aligned}
$$

since $\Phi \cap U^\perp$ is a root system by Lemma 74 as long as it is nonempty.

Now assume $\dim U \geq 2$. Then there exist nonzero subspaces U_1, U_2 of U such that $U = U_1 \oplus U_2$. Then

$$
\begin{aligned}
U_1^\perp \cap U_2^\perp &= (U_1 \oplus U_2)^\perp \\
&= U^\perp.
\end{aligned} \tag{100}
$$

Since $\dim U_1, \dim U_2 < \dim U$, the inductive hypothesis implies

$$
\mathrm{Stab}_W(U_i) = \begin{cases} W(\Phi \cap U_i^\perp) & \text{if } \Phi \cap U_i^\perp \neq \emptyset, \\ \{1\} & \text{otherwise} \end{cases}
$$

for $i = 1, 2$. Suppose first that $\Phi \cap U_1^\perp = \emptyset$. Then $\Phi \cap U^\perp = \emptyset$, and

$$\begin{aligned} \mathrm{Stab}_W(U) &\subset \mathrm{Stab}_W(U_1) \\ &= \{1\}. \end{aligned}$$

Next suppose that $\Phi \cap U_1^\perp \neq \emptyset$. Then

$$\begin{aligned} \mathrm{Stab}_W(U) &= \mathrm{Stab}_W(U_1) \cap \mathrm{Stab}_W(U_2) \\ &= W(\Phi \cap U_1^\perp) \cap \mathrm{Stab}_W(U_2) \\ &= \mathrm{Stab}_{W(\Phi \cap U_1^\perp)}(U_2) \\ &= \begin{cases} W(\Phi \cap U_1^\perp \cap U_2^\perp) & \text{if } \Phi \cap U_1^\perp \cap U_2^\perp \neq \emptyset, \\ \{1\} & \text{otherwise} \end{cases} \\ &= \begin{cases} W(\Phi \cap U^\perp) & \text{if } \Phi \cap U^\perp \neq \emptyset, \\ \{1\} & \text{otherwise} \end{cases} \qquad \text{(by (100))}. \end{aligned}$$

□

Proposition 76. *If U is a subset of V, then*

$$\mathrm{Stab}_W(U) = \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha \in \mathrm{Stab}_W(U) \rangle.$$

Proof. Replacing U by its span, we may assume without loss of generality U is a linear subspace of V. Then by Lemma 75, we have

$$\begin{aligned} \mathrm{Stab}_W(U) &= \begin{cases} W(\Phi \cap U^\perp) & \text{if } \Phi \cap U^\perp \neq \emptyset, \\ \{1\} & \text{otherwise} \end{cases} \\ &= \langle s_\alpha \mid \alpha \in \Phi \cap U^\perp \rangle \\ &= \langle s_\alpha \mid \alpha \in \Phi,\ \forall \lambda \in U,\ (\alpha, \lambda) = 0 \rangle \\ &= \langle s_\alpha \mid \alpha \in \Phi,\ \forall \lambda \in U,\ s_\alpha \lambda = \lambda \rangle \\ &= \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha \in \mathrm{Stab}_W(U) \rangle. \end{aligned}$$

□

Definition 77. The members of the family

$$\{wC \mid w \in W\}$$

are called *chambers*.

Lemma 78. *Let $\Pi = \Phi \cap \mathbf{R}_{\geq 0}\Delta$ be the unique positive system containing Δ. Then*

$$C = \bigcap_{\alpha \in \Pi} H_\alpha^+. \tag{101}$$

In particular,

$$C \subset V \setminus \bigcup_{\beta \in \Phi} H_\beta. \tag{102}$$

Proof. If $\lambda \in C$, then $(\lambda, \alpha) > 0$ for all $\alpha \in \Delta$. Since $\Phi \subset (\mathbf{R}_{\geq 0}\Delta) \cup (\mathbf{R}_{\leq 0}\Delta) \setminus \{0\}$, we see that $(\lambda, \beta) > 0$ for all $\beta \in \Pi$. This implies (101). Since $\Phi = \Pi \cup (-\Pi)$, we see that $(\lambda, \beta) \neq 0$ for all $\beta \in \Phi$. This implies $\lambda \notin \bigcup_{\beta\in\Phi} H_\beta$, proving (102). □

Lemma 79. *If $w \in W$ and $wC \cap C \neq \emptyset$, then $w = 1$. In particular, the group W acts simply transitively on the set of chambers.*

Proof. Suppose $w \in W$ satisfies $wC \cap C \neq \emptyset$. Then there exists $\lambda, \mu \in C$ such that $w\lambda = \mu$. This implies $\{\lambda, \mu\} \subset W\lambda \cap C \subset W\lambda \cap D$. By Theorem 71, we conclude $\lambda = \mu$. This also implies $w \in \mathrm{Stab}_W(\{\lambda\})$, hence $w = 1$ by Lemma 70(iii). In particular, $wC = C$ implies $w = 1$. This shows that W acts simply transitively on the set of chambers. □

Proposition 80.

$$V \setminus \bigcup_{\alpha\in\Phi} H_\alpha = \bigcup_{w\in W} wC \quad \textit{(disjoint).}$$

Proof. By Lemma 79, the chambers are disjoint from each other. Observe

$$\begin{aligned} wC &\subset V \setminus w \bigcup_{\alpha\in\Phi} H_\alpha && \text{(by Lemma 78)} \\ &= V \setminus \bigcup_{\alpha\in\Phi} H_\alpha && \text{(by (99)).} \end{aligned}$$

Thus

$$V \setminus \bigcup_{\alpha\in\Phi} H_\alpha \supset \bigcup_{w\in W} wC \quad \text{(disjoint).}$$

Conversely, let $\lambda \in V \setminus \bigcup_{\alpha\in\Phi} H_\alpha$. By Theorem 71, there exists $w \in W$ such that $w\lambda \in D$, or equivalently, $\lambda \in w^{-1}D$. We claim $\lambda \in w^{-1}C$. Indeed, if $\lambda \notin w^{-1}C$, then

$$\begin{aligned} w\lambda &\in D \setminus C \\ &= \{\mu \in V \mid (\mu, \alpha) \geq 0 \ (\forall \alpha \in \Delta),\ (\mu, \beta) \leq 0 \ (\exists \beta \in \Delta)\} \\ &\subset \{\mu \in V \mid (\mu, \beta) = 0 \ (\exists \beta \in \Delta)\} \\ &= \bigcup_{\beta\in\Delta} H_\beta \\ &\subset \bigcup_{\beta\in\Phi} H_\beta \\ &= w \bigcup_{\beta\in\Phi} H_\beta && \text{(by (99)).} \end{aligned}$$

This implies $\lambda \in \bigcup_{\beta\in\Phi} H_\beta$ which is absurd. This proves the claim, and hence

$$V \setminus \bigcup_{\alpha\in\Phi} H_\alpha \subset \bigcup_{w\in W} wC.$$

□