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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Let Φ be a root system in V, and let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$. Fix a simple system Δ in Φ.

Definition 81. Let $\alpha \in \Phi$ and $w \in W$. The hyperplane H_α is called a *wall* of a chamber wC if $\alpha \in w\Delta$.

Notation 82. For $\lambda \in V$ and $\varepsilon > 0$, denote by $B(\lambda, \varepsilon)$ the ε-ball centered at λ:

$$B(\lambda, \varepsilon) = \{\lambda + \mu \mid \mu \in V,\ \|\mu\| < \varepsilon\}.$$

Lemma 83. *Let $\lambda \in V$ and $\varepsilon > 0$. If w is an orthogonal transformation of V, then $wB(\lambda, \varepsilon) = B(w\lambda, \varepsilon)$.*

Proof.

$$\begin{aligned} wB(\lambda, \varepsilon) &= \{w(\lambda + \mu) \mid \mu \in V,\ \|\mu\| < \varepsilon\} \\ &= \{w\lambda + w\mu \mid \mu \in V,\ \|w\mu\| < \varepsilon\} \\ &= \{w\lambda + \mu \mid \mu \in V,\ \|\mu\| < \varepsilon\} \\ &= B(w\lambda, \varepsilon). \end{aligned}$$

□

Lemma 84. *Let $\alpha \in \Phi$ and $\lambda \in H_\alpha^+$. Then there exists $\varepsilon > 0$ such that $B(\lambda, \varepsilon) \subset H_\alpha^+$.*

Proof. Since $\lambda \in H_\alpha^+$, we have $(\lambda, \alpha) > 0$. Set

$$\varepsilon = \frac{(\lambda, \alpha)}{2\|\alpha\|}.$$

Then for $\mu \in V$ with $\|\mu\| < \varepsilon$, we have

$$\begin{aligned} (\lambda + \mu, \alpha) &= (\lambda, \alpha) + (\mu, \alpha) \\ &\geq (\lambda, \alpha) - |(\mu, \alpha)| \\ &\geq (\lambda, \alpha) - \|\mu\|\|\alpha\| \\ &> (\lambda, \alpha) - \varepsilon\|\alpha\| \\ &= \frac{(\lambda, \alpha)}{2} \\ &> 0. \end{aligned}$$

Thus $\lambda + \mu \in H_\alpha^+$. This implies $B(\lambda, \varepsilon) \subset H_\alpha^+$. □

Lemma 85. *Let $\alpha \in \Phi$ and $\lambda, \mu \in H_\alpha^+$. Then for $0 \leq t \leq 1$, $t\lambda + (1-t)\mu \in H_\alpha^+$.*

Proof. We have

$$(t\lambda + (1-t)\mu, \alpha) = t(\lambda, \alpha) + (1-t)(\mu, \alpha) > 0.$$

□

Proposition 86. *For $\alpha \in \Phi$ and $w \in W$, the following are equivalent:*

(i) *H_α is a wall of wC,*

(ii) *there exist $\lambda \in H_\alpha$ and $\varepsilon > 0$ such that $H_\alpha \cap B(\lambda, \varepsilon) \subset wD$.*

Proof. First we prove the assertion for $w = 1$. Suppose H_α is a wall of C. Then $\alpha \in \Delta$. Then by Lemma 34,

$$s_\alpha(\Pi \setminus \{\alpha\}) = \Pi \setminus \{\alpha\}. \tag{103}$$

Let

$$C' = \bigcap_{\beta \in \Pi \setminus \{\alpha\}} H_\beta^+.$$

Then $C \subset C'$, and

$$\begin{aligned} s_\alpha C &= \bigcap_{\beta \in \Pi} s_\alpha H_\beta^+ \\ &= \bigcap_{\beta \in \Pi} H_{s_\alpha \beta}^+ && \text{(by (97))} \\ &\subset \bigcap_{\beta \in \Pi \setminus \{\alpha\}} H_{s_\alpha \beta}^+ \\ &= \bigcap_{\beta \in s_\alpha(\Pi \setminus \{\alpha\})} H_\beta^+ \\ &= \bigcap_{\beta \in \Pi \setminus \{\alpha\}} H_\beta^+ && \text{(by (103))} \\ &= C'. \end{aligned}$$

Thus

$$C \cup s_\alpha C \subset C'. \tag{104}$$

Let $\lambda_1 \in C$. Then $s_\alpha \lambda_1 \in s_\alpha C$. Set $\lambda = \frac{1}{2}(\lambda_1 + s_\alpha \lambda_1)$. Then $(\lambda, \alpha) = 0$, so $\lambda \in H_\alpha$. Since $\lambda_1, s_\alpha \lambda_1 \in C'$ by (104), Lemma 85 implies $\lambda \in C'$. Then by Lemma 84, for each $\beta \in \Pi \setminus \{\alpha\}$, there exists $\varepsilon_\beta > 0$ such that $B(\lambda, \varepsilon_\beta) \subset H_\beta^+$. Setting

$$\varepsilon = \min\{\varepsilon_\beta \mid \beta \in \Pi \setminus \{\alpha\}\},$$

we obtain $B(\lambda, \varepsilon) \subset C'$. Thus

$$H_\alpha \cap B(\lambda, \varepsilon) \subset H_\alpha \cap C'$$

$$
\begin{aligned}
&= H_\alpha \cap \left(\bigcap_{\beta \in \Pi \setminus \{\alpha\}} H_\beta^+ \right) \\
&\subset (H_\alpha^+ \cup H_\alpha) \cap \left(\bigcap_{\beta \in \Pi \setminus \{\alpha\}} (H_\beta^+ \cup H_\beta) \right) \\
&= D.
\end{aligned}
$$

Conversely, suppose there exist $\lambda \in H_\alpha$ and $\varepsilon > 0$ such that $H_\alpha \cap B(\lambda, \varepsilon) \subset D$. Since $s_\alpha \lambda = \lambda$, we have $s_\alpha B(\lambda, \varepsilon) = B(\lambda, \varepsilon)$ by Lemma 83. This, together with $s_\alpha H_\alpha = H_\alpha$ implies

$$H_\alpha \cap B(\lambda, \varepsilon) \subset s_\alpha D.$$

Thus

$$H_\alpha \cap B(\lambda, \varepsilon) \subset D \cap s_\alpha D. \tag{105}$$

We aim to show $\alpha \in \Delta$. Suppose, by way of contradiction, $\alpha \notin \Delta$. Then $n(s_\alpha) > 1$, so $\Pi \cap s_\alpha(-\Pi) \supsetneqq \{\alpha\}$. This implies that there exists $\beta \in \Pi \setminus \{\alpha\}$ such that $s_\alpha \beta \in -\Pi$. Thus $-s_\alpha \beta \in \Pi$, and hence

$$
\begin{aligned}
D &\subset H_{-s_\alpha\beta}^+ \cup H_{-s_\alpha\beta} \\
&= H_{s_\alpha\beta}^- \cup H_{s_\alpha\beta}.
\end{aligned} \tag{106}
$$

Also, since $\beta \in \Pi$, we have

$$
\begin{aligned}
s_\alpha D &\subset s_\alpha(H_\beta^+ \cup H_\beta) \\
&= H_{s_\alpha\beta}^+ \cup H_{s_\alpha\beta} && \text{(by (96),(97)).}
\end{aligned} \tag{107}
$$

Thus, combining (105)–(107), we find

$$H_\alpha \cap B(\lambda, \varepsilon) \subset H_{s_\alpha\beta}. \tag{108}$$

Since $\beta \neq \pm\alpha$, we have $s_\alpha \beta \neq \pm\alpha$. Thus $H_{s_\alpha\beta} \neq H_\alpha$, which implies that there exists $\mu \in H_\alpha \setminus H_{s_\alpha\beta}$. We may assume $\|\mu\| < \varepsilon$. Then

$$
\begin{aligned}
\lambda + \mu &\in B(\lambda, \varepsilon) \cap H_\alpha \\
&\subset H_{s_\alpha\beta} && \text{(by (108)).}
\end{aligned} \tag{109}
$$

Since

$$
\begin{aligned}
\lambda &\in B(\lambda, \varepsilon) \cap H_\alpha \\
&\subset H_{s_\alpha\beta} && \text{(by (108)),}
\end{aligned}
$$

while $\mu \notin H_{s_\alpha\beta}$, we obtain $\lambda + \mu \notin H_{s_\alpha\beta}$. This contradicts (109).

We have shown that the assertion holds for $w = 1$. We next consider the general case. Let $\alpha \in \Phi$ and $w \in W$. Then

$$
\begin{aligned}
\text{(i)} &\iff \alpha \in w\Delta \\
&\iff w^{-1}\alpha \in \Delta \\
&\iff H_{w^{-1}\alpha} \text{ is a wall of } C \\
&\iff \exists\lambda \in H_{w^{-1}\alpha},\ \exists\varepsilon > 0,\ H_{w^{-1}\alpha} \cap B(\lambda, \varepsilon) \subset D \\
&\iff \exists\lambda \in w^{-1}H_\alpha,\ \exists\varepsilon > 0,\ w^{-1}H_\alpha \cap B(\lambda, \varepsilon) \subset D && \text{(by (96))} \\
&\iff \exists\lambda \in w^{-1}H_\alpha,\ \exists\varepsilon > 0,\ w^{-1}H_\alpha \cap w^{-1}B(w\lambda, \varepsilon) \subset D && \text{(by Lemma 83)} \\
&\iff \exists\mu \in H_\alpha,\ \exists\varepsilon > 0,\ H_\alpha \cap B(\mu, \varepsilon) \subset wD \\
&\iff \text{(ii)}.
\end{aligned}
$$

□

Proposition 87. *If $s \in W$ is a reflection, then there exists $\alpha \in \Phi$ such that $s = s_\alpha$.*

Proof. Since s is a reflection, s fixes a hyperplane H. Let $H^\perp = \mathbf{R}\beta$, where $0 \neq \beta \in V$. Then $s = s_\beta$. Since $s \in \mathrm{Stab}_W(H)$, we have

$$
\begin{aligned}
\{1\} &\neq \mathrm{Stab}_W(H) \\
&= \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha \in \mathrm{Stab}_W(H) \rangle && \text{(by Proposition 76).}
\end{aligned}
$$

This implies that there exists $\alpha \in \Phi$ such that $s_\alpha \in \mathrm{Stab}_W(H)$. The latter implies $s_\alpha = s_\beta = s$. □

Note that Proposition 15 implies that the mapping which sends a root system to a reflection group is a surjection, the following proposition implies that it is essentially an injection.

Proposition 88. *If Φ and Φ' are root systems in V such that $W(\Phi) = W(\Phi')$, then*

$$\{H_\alpha \mid \alpha \in \Phi\} = \{H_{\alpha'} \mid \alpha' \in \Phi'\},$$

or equivalently,

$$\{\mathbf{R}\alpha \mid \alpha \in \Phi\} = \{\mathbf{R}\alpha' \mid \alpha' \in \Phi'\}.$$

Proof. If $\alpha \in \Phi$, then s_α is a reflection in $W(\Phi) = W(\Phi')$. By Proposition 87, there exists $\alpha' \in \Phi'$ such that $s_\alpha = s_{\alpha'}$. This implies $H_\alpha = H_{\alpha'}$. Therefore, we have shown

$$\{H_\alpha \mid \alpha \in \Phi\} \subset \{H_{\alpha'} \mid \alpha' \in \Phi'\}.$$

The reverse containment can be shown in a similar manner. □