August 1, 2016

For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product.

Lemma 1. *For $t \in O(V)$ and $0 \neq \alpha \in V$, we have $ts_\alpha t^{-1} = s_{t\alpha}$.*

Definition 2. Let V be a finite-dimensional vector space over $\mathbf{R}$ with positive definite inner product. Let $W \subset O(V)$ be a finite reflection group. We say that W is *not essential* if there exists a nonzero vector $\lambda \in V$ such that $t\lambda = \lambda$ for all $t \in W$. Otherwise, we say that W is *essential*.

Definition 3. Let Δ be a subset of a root system Φ. We call Δ a *simple system* if Δ is a basis of the subspace spanned by Φ, and if moreover each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign (all nonnegative or all nonpositive). In other words,

$$\Phi \subset \mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta, \tag{1}$$

where

$$\mathbf{R}_{\geq 0}\Delta = \{\sum_{\alpha \in \Delta} c_\alpha \alpha \mid c_\alpha \geq 0 \ (\alpha \in \Delta)\}.$$

If Δ is a simple system, we call its elements *simple roots*.

Theorem 4. *If Δ is a simple system in a root system Φ, then $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$.*

Notation 5. Let $S = \{s_\alpha \mid \alpha \in \Delta\}$. For $I \subset S$, we define

$$\begin{aligned} W_I &= \langle I \rangle, \\ \Delta_I &= \{\alpha \in \Delta \mid s_\alpha \in I\}, \\ V_I &= \mathbf{R}\Delta_I, \\ \Phi_I &= \Phi \cap V_I, \\ \Pi_I &= \Pi \cap V_I. \end{aligned}$$

Proposition 6. *Let $I \subset S$.*

(i) *Φ_I is a root system with simple system Δ_I.*

(ii) *Π_I is the unique positive system of Φ_I containing the simple system Δ_I.*

(iii) *$W(\Phi_I) = W_I$.*

(iv) *Let ℓ be the length function of W with respect to Δ. Then the restriction of ℓ to W_I coincides with the length function ℓ_I of W_I with respect to the simple system Δ_I.*

Proposition 7. *If $s \in W$ is a reflection, then there exists $\alpha \in \Phi$ such that $s = s_\alpha$.*

Proposition 8. *If Φ and Φ' are root systems in V such that $W(\Phi) = W(\Phi')$, then*

$$\{H_\alpha \mid \alpha \in \Phi\} = \{H_{\alpha'} \mid \alpha' \in \Phi'\},$$

or equivalently,

$$\{\mathbf{R}\alpha \mid \alpha \in \Phi\} = \{\mathbf{R}\alpha' \mid \alpha' \in \Phi'\}.$$