CHARACTERIZATIONS FOR CONCAVE FUNCTIONS AND INTEGRAL REPRESENTATIONS

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ABSTRACT. There are several types of concave functions commonly known, which map the unit disk onto a concave domain. Detailed proofs for some familiar inequalities as well as integral representations for these functions and an application concerning the residue are the matter of this article.
Key words: concave univalent functions, integral representations

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane, $\widehat{\mathbb{C}}$ the Riemann sphere and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk. In this article we consider univalent functions, which map $\mathbb{D}$ conformally onto a simply connected domain in $\widehat{\mathbb{C}}$. A univalent function $f : \mathbb{D} \to \widehat{\mathbb{C}}$ is said to be *concave*, if $f(\mathbb{D})$ is *concave*, i.e. $\mathbb{C}\backslash f(\mathbb{D})$ is convex.

Concerning the characteristics and properties, there are several types of concave functions:

(1) A meromorphic, univalent function f is said to be in the class $\mathcal{C}o_0$, if it is concave, has a simple pole at the origin and the representation $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$.
(2) A meromorphic, univalent function f is said to be in the class $\mathcal{C}o_p$ for $p \in (0, 1)$, if it is concave and has a simple pole at p. The normalization for this class can be done by use of the Taylor series expansion at the origin with $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$.
(3) An analytic, univalent function f is said to be in the class $\mathcal{C}o(\alpha)$, if it is concave, satisfies $f(1) = \infty$ with the representation $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ around the origin and an opening angle of $f(\mathbb{D})$ at ∞ less than or equal to $\alpha\pi$ with $\alpha \in (1, 2]$.

It is a well known fact, that the inequality

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}$$

characterizes *convex* functions, mapping the unit disk onto a convex domain. Due to the similarity, the inequality

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) < 0, \quad z \in \mathbb{D}$$

is used - sometimes also as a definition - for concave functions $f \in \mathcal{C}o_0$ (see e.g. [5] and others). Since a complete proof for this statement could not be found in the literature, we are going to present the details in Section 2. Adaptations for the other classes considered in this article were discussed e.g. by Miller in [4] and by Livingston in [3].

Using the given inequalities, several integral representations can be deduced for concave functions. This was first analyzed by Pfaltzgraff and Pinchuk [5], who stated the following for $\mathcal{C}o_0$.

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Concave functions; Integral representations.

Theorem 1.1. [5] *Let* $f : \mathbb{D} \to \widehat{\mathbb{C}}, f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ *be a meromorphic function. Then* $f \in \mathcal{C}o_0$*, if and only if there exists a positive measure* $\mu(t)$ *with* $\int_{-\pi}^{\pi} d\mu(t) = 1$ *and* $\int_{-\pi}^{\pi} e^{-it} d\mu(t) = 0$*, such that for* $z \in \mathbb{D}$

$$f'(z) = -\frac{1}{z^2} \exp \int_{-\pi}^{\pi} 2 \log(1 - e^{-it} z) d\mu(t). \tag{1.1}$$

They used this expression in combination with a linear transformation T, to obtain a characterization for concave functions with pole at $p \in (0, 1)$.

Theorem 1.2. [5] *For* $p \in (0, 1)$*,* $f \in \mathcal{C}o_p$ *if and only if there exists a positive measure* $\mu(t)$ *with* $\int_{-\pi}^{\pi} d\mu(t) = 1$ *and* $\int_{-\pi}^{\pi} T(e^{it}) d\mu(t) = 0$*, such that for* $z \in \mathbb{D}$

$$f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_{-\pi}^{\pi} 2 \log(1 - e^{-it} z) d\mu(t). \tag{1.2}$$

We are going to show different representations for these classes in Section 3, which avoid the use of logarithms and measures.

An analysis for functions in $\mathcal{C}o(\alpha)$ was done by Avkhadiev and Wirths in [1]. The connection to an inequality was discussed by Cruz and Pommerenke [2]. This paper will also deal with the remaining integral representation, using methods presented by Pfaltzgraff and Pinchuk.

As an application of the presented theorems, we will prove the following formula for residues of functions in $\mathcal{C}o_p$.

Theorem 1.3. *Let* $f(z) \in \mathcal{C}o_p$ *be a concave function with a simple pole at some point* $p \in (0, 1)$*. Then the residue of this function* f *can be described by some function* $\varphi : \mathbb{D} \to \mathbb{D}$*, holomorphic in* $\mathbb{D}$ *and* $\varphi(p) = p$*, such that*

$$\operatorname{Res}_p f = -\frac{p^2}{(1-p^2)^2} \exp \int_0^p \frac{-2\varphi(x)}{1 - x\varphi(x)} dx. \tag{1.3}$$

A proof of this theorem will be given in Section 4, as well as some further analysis.

2. Characterizations for concave functions

In this section, we are going to present a variety of characterizations for the different types of concave functions introduced in Section 1.

At first we consider functions in the class $\mathcal{C}o_0$, where the pole lies at the origin.

Theorem 2.1. *Let* $f : \mathbb{D} \to \widehat{\mathbb{C}}$*,* $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ *be a meromorphic function. The function* f *is of class* $\mathcal{C}o_0$*, if and only if the inequality*

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) < 0 \tag{2.1}$$

holds for every $z \in \mathbb{D}$.

A rough idea can be found in [7, p.47]. However, since the details are not carried out, we give a complete presentation of the proof.

First we need the following Lemma.

Lemma 2.2. *Let* $\Delta := \{z \in \widehat{\mathbb{C}} : 1 < |z|\}$ *be the exterior of the unit circle and* $f : \Delta \to \widehat{\mathbb{C}}$ *be a meromorphic univalent function, mapping* Δ *onto the outside of a bounded Jordan curve* Γ *and* $\infty \mapsto \infty$*. This curve* Γ *is analytic, if and only if* f *is analytic and univalent for* $\{z \in \mathbb{C} : r < |z|\}$ *with some* $r < 1$.

A similar statement can be found in [6, p.41] and the construction of the proof goes accordingly.

Proof. If f is analytic and univalent in $\{z \in \mathbb{C} : r < |z|\}$, the curve Γ is obviously analytic. Therefore, let Γ be analytic. Then there exists a univalent function $\varphi : \{z \in \mathbb{C} : \rho < |z| < \frac{1}{\rho}\} \to \mathbb{C}$ with $\rho < 1$, such that $\varphi(\partial\mathbb{D}) = \Gamma$. Furthermore, there exists an $r < 1$, so that $h := \varphi^{-1} \circ f$ is univalent in $\{z \in \mathbb{C} : 1 < |z| < \frac{1}{r}\}$ and $1 < |h(z)| < \frac{1}{\rho}$. Since $|h(z)| \to 1$ as $|z| \to 1$, we can apply the reflexion principle and it follows, that f can be extended to a holomorphic function on $r < |z| < \frac{1}{r}$, where $\rho < |h(z)| < \frac{1}{\rho}$ is satisfied. Thus $f = \varphi \circ h$ is holomorphic on $\{z \in \mathbb{C} : r < |z| < \frac{1}{r}\}$ and therefore analytic and univalent on $r < |z|$. ☐

Proof of Theorem 2.1. We may assume that the nonempty compact convex set $\mathbb{C}\backslash\tilde{f}(\Delta)$ is not a line segment, since otherwise the theorem is trivial. Then $\mathbb{C}\backslash\tilde{f}(\Delta)$ is a convex closed Jordan domain bounded by a simple closed curve.

Let $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \in \mathcal{C}o_0$ for $z \in \mathbb{D}$. Applying the transformation $u : \Delta \to \mathbb{D}$, $z \mapsto \frac{1}{z}$ and setting $\tilde{f} := f \circ u$, we get a concave function, which maps Δ conformally onto the concave domain $f(\mathbb{D})\backslash\{\infty\}$. Therefore there exists a convex domain $G = \mathbb{C}\backslash\overline{\tilde{f}(\Delta)}$, a curve $\Gamma = \partial G$ and a convex function $g : \mathbb{D} \to \mathrm{Int}(\Gamma)$ by use of the Riemann mapping theorem. The curves $\Gamma_k = \{g(z) : |z| = 1 - \frac{1}{k}\}$, $k = 2, 3, \ldots$ are analytic and convex because of the properties of g.

Now let $\tilde{f}_k$ be the functions, which map Δ onto $\mathrm{Ext}(\Gamma_k)$, such that $\tilde{f}_k(\infty) = \infty$ and $\tilde{f}_k'(\infty) > 0$. Due to the definition of Γ_k and Lemma 2.2, each curve can also be described by $\tilde{f}_k(e^{i\vartheta})$ with $\vartheta \in [0, 2\pi)$. Since the interior of the curve Γ_k is convex, $\arg\left(\tilde{f}_k(e^{it}) - \tilde{f}_k(e^{i\vartheta})\right)$ is non-decreasing for $t \in (\vartheta, \vartheta + 2\pi)$. Therefore

$$\begin{aligned} \partial_t \arg\left(\tilde{f}_k(e^{it}) - \tilde{f}_k(e^{i\vartheta})\right) &= \partial_t \mathrm{Im}\log\left(\tilde{f}_k(e^{it}) - \tilde{f}_k(e^{i\vartheta})\right) \\ &= \mathrm{Im}\frac{ie^{it}\tilde{f}_k'(e^{it})}{\tilde{f}_k(e^{it}) - \tilde{f}_k(e^{i\vartheta})} \\ &= \mathrm{Re}\frac{z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} \quad \geq 0 \end{aligned} \tag{2.2}$$

for $z = e^{it} \neq e^{i\vartheta} = \zeta$ and

$$\mathrm{Re}\frac{\zeta + z}{\zeta - z} = \mathrm{Re}\frac{e^{i\vartheta} + e^{it}}{e^{i\vartheta} - e^{it}} = \mathrm{Re}\frac{1 + e^{i(t-\vartheta)}}{1 - e^{i(t-\vartheta)}} = 0 \tag{2.3}$$

holds for the given z, ζ, t and ϑ.

Using the Taylor series expansion $\tilde{f}_k(\zeta) = \sum_{n=0}^{\infty} \frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^n$ for $z, \zeta \in \overline{\Delta}$, we obtain

$$\begin{aligned} \frac{2z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} + \frac{\zeta + z}{\zeta - z} &= \frac{2z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} + \frac{2z}{\zeta - z} + 1 \\ &= 1 + z\frac{2\left(\tilde{f}_k'(z)(\zeta - z) + \tilde{f}_k(z) - \tilde{f}_k(\zeta)\right)}{(\tilde{f}_k(z) - \tilde{f}_k(\zeta))(\zeta - z)} \\ &= 1 + z\frac{-\tilde{f}_k''(z)(\zeta - z)^2 - 2\sum_{n=3}^{\infty} \frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^n}{-\tilde{f}_k'(z)(\zeta - z)^2 - \sum_{n=2}^{\infty} \frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^{n+1}} \end{aligned}$$

$$= \quad 1 + z\frac{\tilde{f}_k''(z) + 2\sum_{n=3}^{\infty}\frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^{n-2}}{\tilde{f}_k'(z) + \sum_{n=2}^{\infty}\frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^{n-1}}.$$

Since

$$\lim_{\zeta\to z}\left(\frac{2z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} + \frac{\zeta + z}{\zeta - z}\right) = 1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)}$$

$\zeta = z$ is a removable singularity.
From (2.2) and (2.3) we obtain

$$\mathrm{Re}\left(\frac{2z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} + \frac{\zeta + z}{\zeta - z}\right) \geq 0$$

for all $|z| = |\zeta| = 1$. Applying the maximum principle first for $|z| > 1$ and then for $|\zeta| > 1$ gives

$$\mathrm{Re}\left(1 + \frac{z\tilde{f}_k''(z)}{\tilde{f}_k'(z)}\right) > 0$$

for all $z \in \Delta$ and $k = 2, 3, \ldots$.

Since convex curves Γ_k converge to Γ for $k \to \infty$, $\tilde{f}_k$ converges locally uniformly to $\tilde{f}$ in Δ due to the kernel theorem of Carathéodory. Therefore

$$\mathrm{Re}\left(1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)}\right) > 0 \tag{2.4}$$

for all $z \in \Delta$.

Considering $\tilde{f} = f \circ u$ with $\tilde{f}'(z) = -\frac{1}{z^2}f'(u)$ and $\tilde{f}''(z) = \frac{1}{z^4}f''(u) + \frac{2}{z^3}f'(u)$, we obtain

$$\begin{aligned} 1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)} &= 1 + \frac{z\left(\frac{1}{z^4}f''(u) + \frac{2}{z^3}f'(u)\right)}{-\frac{1}{z^4}f'(u)} \\ &= 1 - \frac{\frac{1}{z}f''(u) + 2f'(u)}{f'(u)} \\ &= -1 - \frac{uf''(u)}{f'(u)}, \qquad (2.5) \end{aligned}$$

hence (2.1).

The second implication is the same as for the convex case, when one considers $z \in \Delta$. This can be found in various textbooks, see e.g. [7]. Applying transformation (2.5) yields the statement. □

Remark 2.3. It is also

$$\begin{aligned} \frac{2z\tilde{f}'(z)}{\tilde{f}(z) - \tilde{f}(\zeta)} + \frac{\zeta + z}{\zeta - z} &= \frac{2z\frac{1}{z^2}\tilde{f}'(u(z))}{f(u(z)) - f(u(\zeta))} + \frac{\frac{1}{u(\zeta)} + \frac{1}{u(z)}}{\frac{1}{u(\zeta)} - \frac{1}{u(z)}} \\ &= -\frac{2u(z)f'(u(z))}{f(u(z)) - f(u(\zeta))} - \frac{u(\zeta) + u(z)}{u(\zeta) - u(z)} \end{aligned}$$

for $z, \zeta \in \Delta$. Therefore we obtain

$$\mathrm{Re}\left(\frac{2zf'(z)}{f(z) - f(\zeta)} + \frac{\zeta + z}{\zeta - z}\right) \leq 0 \tag{2.6}$$

for $f \in \mathcal{C}o_0$ and $z, \zeta \in \mathbb{D}$ from the proof of Theorem 2.1.

Livingston [3] adapted the characterization for functions in $\mathcal{C}o_0$ for the class of concave functions with pole at $p \in (0,1)$, using the transformation $z \mapsto \frac{z+p}{1+pz}$.

Theorem 2.4. [3] *Let $p \in (0,1)$ and f be a meromorphic function. It is $f \in \mathcal{C}o_p$, if and only if*

$$\operatorname{Re}\left(1 + p^2 - 2pz + \frac{(z-p)(1-pz)f''(z)}{f'(z)}\right) < 0 \tag{2.7}$$

for $z \in \mathbb{D}$.

From this theorem, we can obtain a statement similar to Remark 2.3 for functions in $\mathcal{C}o_p$. It should be mentioned, that Theorem 2.4 and the following statements are valid regardless of the normalization introduced for this class in Section 1.

Corollary 2.5. *Let $p \in (0,1)$, $f \in \mathcal{C}o_p$ and $z, \zeta \in \mathbb{D}$. Then*

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z)-f(\zeta)} + \frac{\zeta+z}{\zeta-z} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right) < 0. \tag{2.8}$$

Originally this was proved by Miller in [4]. Considering Livingston's analysis in [3], we can give an alternate proof.

Proof. Let

$$P(z) := -1 - p^2 + 2pz - 2(z-p)(1-pz)\frac{f'(z)(\zeta-z) + f(z) - f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}. \tag{2.9}$$

Using $f(z) = \sum_{n=-1}^{\infty} b_n (z-p)^n$ for $\zeta \neq p$ and

$$\begin{aligned}(z-p)\frac{f'(z)}{f(z)-f(\zeta)} &= \frac{[-b_{-1}(z-p)^{-1} + b_1(z-p) + \cdots]}{b_{-1}(z-p)^{-1} + b_0 + \cdots - f(\zeta)}\\ &= \frac{[-b_{-1} + b_1(z-p)^2 + \cdots]}{b_{-1} + b_0(z-p) + \cdots - f(\zeta)(z-p)}\end{aligned}$$

we have

$$P(p) = -1 - p^2 + 2p^2 - 2(1-p^2)\frac{-b_{-1}(\zeta-p)}{b_{-1}(\zeta-p)} = 1 - p^2.$$

Furthermore, observing that

$$\begin{aligned}zP(z) + pz^2 - p &= 3pz^2 - z - p^2z - p\\ &\qquad - 2z(z-p)(1-pz)\frac{f'(z)(\zeta-z) + f(z) - f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}\\ \Leftrightarrow \quad \frac{zP(z) + pz^2 - p}{(z-p)(1-pz)} &= \frac{2pz^2 - 2p^2z + p^2z - z + pz^2 - p}{(z-p)(1-pz)}\\ &\qquad - 2z\frac{f'(z)(\zeta-z) + f(z) - f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}\\ &= \frac{2pz}{1-pz} - \frac{z+p}{z-p} - 2z\frac{f'(z)(\zeta-z) + f(z) - f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}\\ &= \frac{1+pz}{1-pz} - \frac{z+p}{z-p} - 1 - 2z\frac{f'(z)(\zeta-z) + f(z) - f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}\\ &= \frac{1+pz}{1-pz} - \frac{z+p}{z-p} - \frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{\zeta+z}{\zeta-z} \qquad (2.10)\end{aligned}$$

and defining

$$Q(z) := \frac{zP(z) + pz^2 - p}{(z-p)(1-pz)}, \tag{2.11}$$

we obtain $Q(p) = \frac{1+p^2}{1-p^2}$ and

$$\lim_{\zeta \to z} Q(z) = -1 - \frac{zf''(z)}{f'(z)} + \frac{1+pz}{1-pz} - \frac{z+p}{z-p}.$$

Therefore the function

$$F(z,\zeta) = \begin{cases} 1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{z+p}{z-p} - \dfrac{1+pz}{1-pz}, & \text{for } z = \zeta \\ \dfrac{2zf'(z)}{f(z)-f(\zeta)} + \dfrac{\zeta+z}{\zeta-z} + \dfrac{z+p}{z-p} - \dfrac{1+pz}{1-pz}, & \text{for } z \neq \zeta \end{cases} \tag{2.12}$$

is holomorphic for z, $\zeta \in \mathbb{D}$.

Considering Theorem 2.1, Remark 2.3 and the fact that

$$\operatorname{Re}\left(\frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right) = 0,$$

we obtain (2.8) by the maximum principle. □

The case $z = \zeta$ in (2.12) was deduced by different means by Pfaltzgraff and Pinchuk in [5]. $\operatorname{Re} F(z,z) < 0$ for $z \in \mathbb{D}$ also holds as a necessary and sufficient condition for a meromorphic function f to be in $\mathcal{C}o_p$.

Theorem 2.6. [5] *Let $f : \mathbb{D} \to \widehat{\mathbb{C}}$ be a meromorphic function. The function f is of class $\mathcal{C}o_p$, if and only if for $z \in \mathbb{D}$*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right) < 0. \tag{2.13}$$

For the class $\mathcal{C}o(\alpha)$, the following inequality can be deduced.

Theorem 2.7. [1, 2] *Let $\alpha\pi$, $\alpha \in (1,2]$. An analytic function f with $f(0) = f'(0) - 1 = 0$ is of class $\mathcal{C}o(\alpha)$, if and only if for $z \in \mathbb{D}$*

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)} - \frac{\alpha+1}{2} \cdot \frac{1+z}{1-z}\right) < 0. \tag{2.14}$$

Avkhadiev and Wirths considered this in [1] and Cruz and Pommerenke discussed a variation of the theorem in detail in [2]. A factor $\frac{2}{\alpha-1}$ has to be added to the characterization in case a normalization is required.

3. Integral representations for concave functions

The inequalities from the previous section provide new representation formulas for concave functions. These are equivalent to the already presented characterizations.

Theorem 3.1. *Let $f : \mathbb{D} \to \widehat{\mathbb{C}}, f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ be a meromorphic function. If $f \in \mathcal{C}o_0$, then a function $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic in $\mathbb{D}$ with $\varphi(0) = 0$ exists, such that for $z \in \mathbb{D}$*

$$f'(z) = -\frac{1}{z^2} \exp \int_0^z \frac{-2\,\varphi(\zeta)}{1 - \zeta\varphi(\zeta)} d\zeta. \tag{3.1}$$

On the other hand, for any holomorphic function φ mapping $\mathbb{D} \to \mathbb{D}$ with $\varphi(0) = 0$, there exists a function $f \in \mathcal{C}o_0$ described by (3.1).

Proof. It is known, that a function which maps $\mathbb{D}$ into the right half plane and the origin to 1 can be expressed as $\frac{1+z\varphi(z)}{1-z\varphi(z)}$, where $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$ is a function holomorphic in $\mathbb{D}$. We combine this fact with Theorem 2.1. Therefore there exists a holomorphic function φ with the given properties such that

$$1 + z\frac{f''(z)}{f'(z)} = -\frac{1+z\varphi(z)}{1-z\varphi(z)}.$$

Hence it is also

$$\begin{aligned} && 2 + z\frac{f''(z)}{f'(z)} &= -\frac{2z\varphi(z)}{1-z\varphi(z)} \\ \Leftrightarrow && \frac{2}{z} + \frac{f''(z)}{f'(z)} &= \frac{-2\varphi(z)}{1-z\varphi(z)} \\ \Leftrightarrow && \frac{d}{dz}\log\left(-z^2 f'(z)\right) &= \frac{-2\varphi(z)}{1-z\varphi(z)} \\ \Leftrightarrow && \log\left(-z^2 f'(z)\right) - \log\left(-\zeta^2 f'(\zeta)\right)\big|_{\zeta=0} &= \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta. \end{aligned}$$

Using $\zeta^2 f'(\zeta) = \zeta^2 \cdot \left(\frac{-1}{\zeta^2} + \sum_{n=1}^{\infty} n a_n \zeta^{n-1}\right) = -1 + \mathcal{O}(\zeta^2)$, with $\mathcal{O}$ being the Landau symbol as in the proof of Corollary 2.5, we obtain

$$\begin{aligned} && \log\left(-z^2 f'(z)\right) &= \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta \\ \Leftrightarrow && f'(z) &= -\frac{1}{z^2}\exp\int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta. \end{aligned}$$

Since f' must not have a residue, it has to be

$$\left(z^2\, f'(z)\right)'\Big|_{z=0} = 0.$$

Considering $\kappa(z) := \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta$ and $g(z) := z^2\, f'(z) = -e^{\kappa(z)}$, we obtain

$$\frac{g'(z)}{g(z)} = \kappa'(z) = \frac{\varphi(z)}{1-z\varphi(z)}.$$

Therefore it has to be $\varphi(0) = 0$.

Conversely, if $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with $\varphi(0) = 0$, a function f defined by

$$f'(z) = -\frac{1}{z^2}\exp\int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta$$

does not have a residue of its own. Furthermore we obtain

$$1 + z\frac{f''(z)}{f'(z)} = -\frac{1+z\varphi(z)}{1-z\varphi(z)}.$$

By the use of Theorem 2.1, concavity follows immediately. □

Using the inequality obtained from Theorem 2.6, it is possible to prove a similar statement for the class Co_p.

Theorem 3.2. *Let $p \in (0,1)$. If a meromorphic function $f : \mathbb{D} \to \widehat{\mathbb{C}}$ belongs to the class $\mathcal{C}o_p$, then there exists a function $\varphi : \mathbb{D} \to \mathbb{D}$, holomorphic in $\mathbb{D}$ with $\varphi(p) = p$, such that the concave function can be represented as*

$$f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta \tag{3.2}$$

for $z \in \mathbb{D}$. Conversely, for any holomorphic function φ mapping $\mathbb{D} \to \mathbb{D}$ with $\varphi(p) = p$, there exists a concave function of class $\mathcal{C}o_p$ described by (3.2).

Proof. From Theorem 2.6 it is known, that $f \in \mathcal{C}o_p$ is equivalent to

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+zp}{1-zp}\right) < 0$$

for $p \in (0,1)$. Therefore there exists a function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$ such that

$$\begin{aligned}
& & 1 + z\frac{f''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz} &= -\frac{1+z\varphi(z)}{1-z\varphi(z)}\\
&\Leftrightarrow & 1 + z\frac{f''(z)}{f'(z)} + \left(\frac{2z}{z-p} - 1\right) - \left(\frac{2pz}{1-pz} + 1\right) &= -1 - \frac{2z\varphi(z)}{1-z\varphi(z)}\\
&\Leftrightarrow & z\frac{f''(z)}{f'(z)} + \frac{2z}{z-p} - \frac{2pz}{1-pz} &= -\frac{2z\varphi(z)}{1-z\varphi(z)}\\
&\Leftrightarrow & \frac{f''(z)}{f'(z)} + \frac{2}{z-p} - \frac{2p}{1-pz} &= -\frac{2\varphi(z)}{1-z\varphi(z)}\\
&\Leftrightarrow & \frac{d}{dz}\left(\log(f'(z)) + 2\log(z-p) + 2\log(1-pz)\right) &= -\frac{2\varphi(z)}{1-z\varphi(z)}.
\end{aligned}$$

Integration yields

$$\begin{aligned}
& & \log\left(f'(z)(z-p)^2(1-pz)^2\right) - \log p^2 &= -\,2\int_0^z \frac{\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta\\
&\Leftrightarrow & f'(z) &= \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta.
\end{aligned}$$

Similar to the case of Theorem 3.1, the representation (3.2) must not have a residue of its own because of the properties of $f'(z)$. It has to be

$$\left((z-p)^2 f'(z)\right)'\Big|_{z=p} = 0. \tag{3.3}$$

Setting $\kappa(z) := \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta$ and $g(z) := (z-p)^2 f'(z) = \frac{p^2}{(1-zp)^2} e^{\kappa(z)}$, we obtain

$$\frac{g'(z)}{g(z)} = \kappa'(z) + \frac{2p}{1-zp}.$$

Therefore it is necessary to be

$$\begin{aligned}
& & \frac{-2\varphi(p)}{1-p\varphi(p)} + \frac{2p}{1-p^2} &= 0\\
&\Leftrightarrow & \varphi(p) &= p,
\end{aligned}$$

so that (3.3) is satisfied.

On the other hand, if $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with $\varphi(p) = p$, the function f defined by (3.2) does not have a residue of its own due to the consideration of the above. Furthermore it satisfies

$$1 + z\frac{f''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz} = -\frac{1+z\varphi(z)}{1-z\varphi(z)}.$$

Using Theorem 2.6, we obtain $f \in \mathcal{C}o_p$. □

Considering the class $\mathcal{C}o(\alpha)$ Avkhadiev and Wirths proved the following in [1].

Theorem 3.3. [1] *Let $\alpha \in (1, 2]$ and $f : \mathbb{D} \to \widehat{\mathbb{C}}$ be an analytic function with $f(0) = f'(0) - 1 = 0$. Then $f \in \mathcal{C}o(\alpha)$ if and only if there exists a function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$, such that for $z \in \mathbb{D}$*

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}} \exp \int_0^z -(\alpha-1)\frac{\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta. \tag{3.4}$$

Using a positive measure $\mu(t)$ as in Theorem 1.1, we can obtain the next statement.

Theorem 3.4. *Let $\alpha \in (1, 2]$ and f be an analytic function with $f(0) = f'(0) - 1 = 0$. Then $f \in \mathcal{C}o(\alpha)$ if and only if there exists a positive measure $\mu(t)$ with $\int_{-\pi}^{\pi} \mu(t)dt = 1$, such that*

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}} \exp \int_{-\pi}^{\pi} (\alpha-1) \log(1 - e^{-it}z) d\mu(t). \tag{3.5}$$

Proof. The normalized, analytic function f is of class $\mathcal{C}o(\alpha)$ if and only if (2.14) of Theorem 2.7 is valid. It is known, that every function $P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ with $\operatorname{Re} P(z) > 0$ for $z \in \mathbb{D}$ can be represented as

$$P(z) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t),\ z \in \mathbb{D}$$

with some positive measure $\mu(t)$ due to the Herglotz representation formula.
From the normalized form of (2.14) we therefore obtain the existence of a positive measure $\mu(t)$, with $\int_{-\pi}^{\pi} d\mu(t) = 1$, such that

$$\begin{aligned}
& -\frac{2}{\alpha-1}\left(1 + z\frac{f''(z)}{f'(z)} - \frac{\alpha+1}{2}\cdot\frac{1+z}{1-z}\right) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow\quad & \frac{2}{\alpha-1}\left((\alpha+1)\frac{z}{1-z} + \frac{\alpha-1}{2} - z\frac{f''(z)}{f'(z)}\right) - 1 = \int_{-\pi}^{\pi} \frac{2z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow\quad & \frac{2z(\alpha+1)}{(\alpha-1)(1-z)} - \frac{2}{\alpha-1}\cdot\frac{zf''(z)}{f'(z)} = \int_{-\pi}^{\pi} \frac{2z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow\quad & \frac{\alpha+1}{1-z} z - z\frac{f''(z)}{f'(z)} = (\alpha-1)\int_{-\pi}^{\pi} \frac{z}{e^{it}-z} d\mu(t).
\end{aligned}$$

Considering the derivative leads to

$$\begin{aligned}
& \frac{d}{dz}\left(-(\alpha+1)\log(1-z) - \log f'(z)\right) = -(\alpha-1)\int_{-\pi}^{\pi} \frac{d}{dz}\log(1-e^{it}z) d\mu(t) \\
\Leftrightarrow\quad & \log(1-z)^{\alpha+1} f'(z) = (\alpha-1)\int_{-\pi}^{\pi} \log(1-e^{it}z) d\mu(t),
\end{aligned}$$

which is obviously equivalent to the desired representation formula. □

Since we do not have to deal with any complications concerning the logarithm during the proof of Theorem 3.4, there are no additional conditions for the measure, as it was the case in the previous theorems.

Remark 3.5. As it can easily be observed, there is a similarity between the representation formula using a function φ (see e.g. Theorem 3.1) and the version considering a positive measure $\mu(t)$ (see e.g. Theorem 1.1).

Since the expression $z \mapsto \frac{1+z\varphi(z)}{1-z\varphi(z)}$, with $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$, φ holomorphic in $\mathbb{D}$, maps the unit disk onto the right half of the complex plane and is normalized by $0 \mapsto 1$, it can also be described by means of the Herglotz representation formula. Therefore

$$\begin{aligned}
& & \frac{1+z\varphi(z)}{1-z\varphi(z)} &= \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) \\
&\Leftrightarrow & \frac{2z\varphi(z)}{1-z\varphi(z)} + 1 &= \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) \\
&\Leftrightarrow & \frac{2\varphi(z)}{1-z\varphi(z)} &= \int_{-\pi}^{\pi} \frac{2z}{e^{it}-z} d\mu(t) \\
(3.6) \qquad &\Leftrightarrow & \int_0^z \frac{\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta &= \int_{-\pi}^{\pi} \log(1-e^{it}z) d\mu(t)
\end{aligned}$$

for $z \in \mathbb{D}$. The existence of a certain function φ is hereby equivalent to the existence of a positive measure $\mu(t)$ so that (3.6) holds.

However, since the representation using the measure involves logarithms, we have to be careful with additional conditions, to ensure that the results are well-defined. On the other hand, additional conditions for φ are provided by the fact, that $f'(z)$ must not have a residue of its own, as shown in the previous proofs.

4. Application

Using the integral representation formula of the previous section for the class $\mathcal{C}o_p$, we can obtain Theorem 1.3 for the residue of concave functions.

Proof of Theorem 1.3. Since a function $f \in \mathcal{C}o_p$ is represented by

$$f(z) = \frac{b_{-1}}{z-p} + b_0 + \sum_{n=1}^{\infty} b_n (z-p)^n$$

for $|z-p| < 1-p$, we obtain

$$f'(z) = -\frac{b_{-1}}{(z-p)^2} + b_1 + \sum_{n=2}^{\infty} n\, b_n (z-p)^{n-1}.$$

Applying (3.2) from Theorem 3.2, the following equality is valid.

$$\frac{-b_{-1}}{(z-p)^2} + b_1 + \sum_{n=2}^{\infty} n\, b_n (z-p)^{n-1} = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta.$$

Multiplying both sides with $-(z-p)^2$ we have

$$b_{-1} - b_1(z-p)^2 - \sum_{n=2}^{\infty} n\, b_n (z-p)^{n+1} = \frac{-p^2}{(1-zp)^2} \exp \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta.$$

Considering $z = p$ leads to the theorem. $\square$

Using the inequality

$$\left|\frac{1}{f(z)}-\frac{1}{z}+\frac{1+p^2}{p}\right|\leq 1$$

for $f\in\mathcal{C}o_p$ provided by Miller in [4], Wirths proved the next theorem in [8].

Theorem 4.1. [8] *Let $p\in(0,\,1)$. For $a\in\mathbb{C}$ there exists a function $f\in\mathcal{C}o_p$ such that $a=\operatorname{Res}_p f$ if and only if*

$$\left|a+\frac{p^2}{1-p^4}\right|\leq\frac{p^4}{1-p^4}. \tag{4.1}$$

Let $\vartheta\in[0,\,2\pi)$. A function $f\in\mathcal{C}o_p$ has the residue

$$a=-\frac{p^2}{1-p^4}+e^{i\vartheta}\frac{p^4}{1-p^4}$$

if and only if

$$f_\vartheta(z)=\frac{z-\frac{p}{1+p^2}(1+e^{i\vartheta})z^2}{\left(1-\frac{z}{p}\right)(1-pz)}. \tag{4.2}$$

We now prove the same statement using the given integral representation.

Proof. Applying the transformation $x=\frac{p-z}{1-pz}$ and $\Phi(z)=\varphi(x)$ for $p\in(0,1)$ and $z\in\mathbb{D}$ to (1.3) we obtain

$$\begin{aligned}\int_0^p\frac{-2\varphi(x)}{1-x\varphi(x)}dx&=\int_p^0\frac{-2\Phi(z)}{1-\frac{p-z}{1-pz}\Phi(z)}\cdot\frac{p^2-1}{(1-pz)^2}dz\\&=\int_0^p\frac{2\Phi(z)(p^2-1)}{(1-pz)^2-(p-z)\Phi(z)(1-pz)}dz.\end{aligned}$$

The function $\Phi:\mathbb{D}\to\mathbb{D}$ is holomorphic in $\mathbb{D}$ with $\Phi(0)=p$. Therefore there exists a function $\Psi:\mathbb{D}\to\mathbb{D}$, holomorphic in $\mathbb{D}$ with $\Psi(0)=0$, such that $\Phi(z)=\frac{p-\Psi(z)}{1-p\Psi(z)}$. Then

$$\begin{aligned}\int_0^p\frac{-2\varphi(x)}{1-x\varphi(x)}dx&=\int_0^p\frac{2(p^2-1)(p-\Psi(z))}{(1-pz)\left((1-p^2)-z\Psi(z)(1-p^2)\right)}dz\\&=\int_0^p\frac{2(\Psi(z)-p)}{(1-pz)(1-z\Psi(z))}dz.\end{aligned} \tag{4.3}$$

Using representation (1.3) and (4.3) for the residue , we obtain

$$\begin{aligned}&\left|a+\frac{p^2}{1-p^4}\right|\\&\overset{(1.3)}{=}\left|\frac{p^2}{(1-p^2)^2}\exp\int_0^p\frac{-2\varphi(x)}{1-x\varphi(x)}dx-\frac{p^2}{1-p^4}\right|\\&\overset{(4.3)}{=}\frac{p^2}{1-p^4}\left|\frac{1+p^2}{1-p^2}\exp\int_0^p\frac{2(\Psi(x)-p)}{(1-px)(1-x\Psi(x))}dx-1\right|\\&=\frac{p^2}{1-p^4}\left|\exp\int_0^p\left(\frac{2p}{1-p^2x^2}+\frac{2(\Psi(x)-p)}{(1-px)(1-x\Psi(x))}\right)dx-1\right|\\&=\frac{p^2}{1-p^4}\left|\exp\int_0^p\frac{2(\Psi(x)-p^2x)}{(1-x\Psi(x))(1-p^2x^2)}dx-1\right|.\end{aligned}$$

Furthermore, since $\left|\frac{w-p^2x}{1-xw}\right| \le \frac{(1-p^2)x}{1-x^2}$ for $|w| \le x$ and $|\Psi(x)| \le x$, we have

$$\left|\frac{\Psi(x)-p^2x}{1-x\Psi(x)}\right| \le \frac{(1-p^2)x}{1-x^2} \tag{4.4}$$

for any $x > 0$.

Combining (4.4) with the above and applying $|e^w - 1| \le e^{|w|} - 1$, we obtain

$$\begin{aligned} \left|a + \frac{p^2}{1-p^4}\right| &\le \frac{p^2}{1-p^4}\left(\exp\left|\int_0^p \frac{2(1-p^2)x}{(1-x^2)(1-p^2x^2)}dx\right| - 1\right) \\ &= \frac{p^4}{1-p^4}. \end{aligned} \tag{4.5}$$

Setting $\Psi(z) = \Psi_c(z) = c\,z$, i.e. $\varphi_c(\frac{p-z}{1-pz}) = \frac{p-c\,z}{1-p\,c\,z}$ with $c \in \overline{\mathbb{D}}$ we obtain a concave function, for which equality holds in (4.5) if and only if $c \in \partial\mathbb{D}$.

Applying the transformation $z = \frac{\zeta-p}{p\zeta-1}$ we have

$$\varphi_c(\zeta) = \frac{-(c-p^2)\zeta - (1-c)p}{(1-c)p\,\zeta + p^2c - 1}. \tag{4.6}$$

Inserting (4.6) into the integral representation (3.2) of Theorem 2.6 leads to the concave function g_c which first derivative can be expressed as

$$\begin{aligned} g_c'(z) &= \frac{p^2}{(z-p)^2(1-pz)^2}\exp\int_0^z \frac{2(c-p^2)\zeta + 2(1-c)p}{(c-p^2)\zeta^2 + 2(1-c)p\,\zeta + p^2c - 1}d\zeta \\ &= \frac{p^2}{(z-p)^2(1-pz)^2}\left(\frac{(c-p^2)z^2 + 2(1-c)pz + p^2c - 1}{p^2c-1}\right) \\ &= \frac{p^2}{(z-p)^2(1-pz)^2}\left(\frac{c-p^2}{p^2c-1}z^2 + \frac{1-c}{p^2c-1}2pz + 1\right). \end{aligned}$$

Therefore

$$\begin{aligned} g_c(z) &= \frac{p^2}{p^2c-1}\left[\frac{1}{\zeta-p} + \frac{c}{p(1-\zeta p)}\right]_0^z \\ &= \frac{-p^2}{p^2c-1}\left(\frac{(1-c)pz^2 + (p^2c-1)z}{p(z-p)(1-pz)}\right) \\ &= \frac{z - \frac{p}{1+p^2}\left(1 - \frac{c-p^2}{p^2c-1}\right)z^2}{(1-\frac{z}{p})(1-pz)} \end{aligned} \tag{4.7}$$

for $c \in \overline{\mathbb{D}}$.

Furthermore, the residue of g_c can be determined to be

$$\begin{aligned} \operatorname{Res}_p g_c &= \frac{p}{p^2c-1} \cdot \left.\frac{(c-1)pz^2 + (1-p^2c)z}{1-pz}\right|_{z=p} \\ &= \frac{p^2-p^4}{(p^2c-1)(1-p^2)} = \frac{p^2}{p^2c-1} \end{aligned}$$

for any $c \in \overline{\mathbb{D}}$. Since the image of $\overline{\mathbb{D}}$ under the mapping $z \mapsto \frac{p^2}{p^2z-1}$ is the closed disk K with center $\frac{-p^2}{1-p^4}$ and radius $\frac{p^4}{1-p^4}$, for each number $a \in K$ there exists a concave function g_c with $c = \frac{p^2+a}{p^2a}$ such that $\operatorname{Res}_p g_c = a$.

The extremal function in case of equality can be obtained by considering $c \in \partial\mathbb{D}$ as mentioned above and setting $-\frac{c-p^2}{p^2c-1} = e^{i\vartheta}$ in (4.7) for some $\vartheta \in [0, 2\pi)$. This proves the theorem. $\square$

Acknowledgements: Section 2 of this article was a part of the author's Diplomthesis and he wants to thank Prof. Stephan Ruscheweyh for the suggestions and remarks during that time. The author also likes to thank Prof. Toshiyuki Sugawa for his helpful comments and encouragement concerning this topic and the referee for the careful reading of this paper.

References

[1] F.G. Avkhadiev and K.-J. Wirths, *Concave schlicht functions with bounded opening angle at infinity*, Lobachevskii J. of Math. **17** (2005), 3-10.

[2] L. Cruz and Ch. Pommerenke, *On concave univalent functions*, Complex Var. and Elliptic Equ. **52** (2007), 153-159.

[3] A.E. Livingston, *Convex meromorphic mappings*, Ann. Polo. Math. **59** (1994), 275-291.

[4] J. Miller, *Convex and starlike meromorphic functions*, Proc. Amer. Math. Soc. **80** (1980), 607-613.

[5] J. Pfaltzgraff and B. Pinchuk, *A variational method for classes of meromorphic functions*, J. Analyse Math. **24** (1971), 101-150.

[6] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, 1992.

[7] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Goettingen, 1975.

[8] K.-J. Wirths, *On the residuum of concave univalent functions*, Serdica Math. J. **32** (2006), 209-214.

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