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$\boxed{1}$ Let $a > 0$, $b > 0$, $c > 0$ be given constants. Find the maximum of the function $x^a y^b z^c$ of $x > 0$, $y > 0$, $z > 0$ under the constraint $x^2 + y^2 + z^2 = 1$.

$\boxed{2}$

(1) Let $0 < t < 1$ be fixed. Draw the graph of the function $y = x^{t-1}e^{-x}$ $(x > 0)$.

(2) Prove that the integral $\displaystyle\int_0^1 x^{t-1}e^{-x}dx$ converges for $0 < t < 1$.

(3) Prove that the integral $\displaystyle\int_1^\infty x^{t-1}e^{-x}dx$ converges for $t > 0$.

(4) By (2) and (3), for each $t > 0$, we can define

$$f(t) = \int_0^\infty x^{t-1}e^{-x}dx.$$

Show that the equality $f(t+1) = tf(t)$ holds for any $t > 0$, and find the value $f(5)$.

$\boxed{3}$ Consider the linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$

$$f(\boldsymbol{x}) = A\boldsymbol{x} \qquad (\boldsymbol{x} \in \mathbb{R}^3)$$

determined by a real matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & a & b \\ 2 & b & c \end{pmatrix}.$$

Assume that if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ are orthogonal then so are $f(\boldsymbol{x})$ and $f(\boldsymbol{y})$. Moreover, assume that $a < 0$.

(1) Find (a, b, c).

(2) Diagonalize the matrix A. Find also an orthogonal matrix for the diagonalization.

$\boxed{4}$ Let V be a 4-dimensional real vector space. Let $J : V \to V$ be a linear map satisfying

$$J^2 = -\ I.$$

Here, J^2 means the composed map $J \circ J$, and I represents the indentity map on V.

(1) Show that the linear map $J : V \to V$ is an isomorphism.

(2) Show that v_1 and $J(v_1)$ are linearly independent for any non-zero vector $v_1 \in V$. Moreover, when V_1 is the subspace generated by $\{v_1, J(v_1)\}$, show that V_1 is a J-invariant subspace, i.e.,

$$J(V_1) \subset V_1$$

holds.

(3) Take any vector $v_2 \in V$ which does not belong to V_1, where V_1 is the subspace defined in (2). Let V_2 be the J-invariant subspace generated by $\{v_2, J(v_2)\}$. Then, show that

$$V_1 \cap V_2 = \{\mathbf{0}\},$$

where $\mathbf{0}$ is the zero vector of V.

(4) Find the representation matrix A of J with respect to the basis $\{v_1, J(v_1), v_2, J(v_2)\}$ of V obtained by (2) and (3).

$\boxed{5}$ For each positive integer n, we set $X_n = \{1, 2, \ldots, n\}$. Let E be a family of sets $\{x, y\}$ consisting of two distinct elements x, y of X_n. We say that E has a triangle if there exists a subset $\{x, y, z\}$ of X_n such that $\{x, y\} \in E$, $\{y, z\} \in E$, and $\{x, z\} \in E$. Let e_n denote the maximum of the number of the elements of E which does not have a triangle. For example, $e_3 = 2$.

(1) Find e_4.

(2) Prove that $e_5 \geq 6$ holds.

(3) Prove that $e_n \leq e_{n-2} + n - 1$ holds if $n \geq 5$.

$\boxed{6}$ We toss a coin infinitely many times, assuming that the probability of heads on each toss is p $(0 < p < 1)$. We define a sequence of random variables $Z_1, Z_2, \ldots$ by $Z_n = 1$ if the nth outcome is heads and $Z_n = 0$ if it is tails. We set

$$X_n = Z_1 + Z_2 + \cdots + Z_n \qquad (n = 1, 2, \ldots)$$

and define, for each natural number $k \geq 1$,

$$T_k = \inf\{n \geq 1 \mid X_n \geq k\}.$$

(1) Find the probability $P(T_2 = n)$ for $n = 1, 2, \ldots$.

(2) Find the mean value $\mathbf{E}[T_2]$ of T_2.

(3) Let k, m, n be natural numbers satisfying $k \geq 3$, $n < m$. Find the conditional probability $P(T_k = m | T_2 = n)$ in terms of binomial coefficients.

$\boxed{7}$ Let i be the imaginary unit. We define a meromorphic function f on the complex plane by $f(z) = \dfrac{e^{iz}}{(z^2+1)^2}$.

(1) For $r > 0$, we denote by $M(r)$ the maximum of $|f(re^{i\theta})|$ on the interval $0 \leq \theta \leq \pi$. Show that $\lim\limits_{r \to +\infty} r^2 M(r) = 0$.

(2) Find all poles of $f(z)$ in the upper half-plane $\mathrm{Im}\ z > 0$ and the associated residues at those poles.

(3) Compute the integral $I = \displaystyle\int_0^\infty \frac{\cos x}{(x^2+1)^2}\, dx.$

$\boxed{8}$ Let V be the set of all 2×2 matrices whose components are real-valued continuous functions on $[-\pi, \pi]$, i.e.,

$$V = \left\{ A(\theta) = \begin{pmatrix} a_{11}(\theta) & a_{12}(\theta) \\ a_{21}(\theta) & a_{22}(\theta) \end{pmatrix} : \right.$$
$$\left. a_{ij}\ (i, j = 1, 2) \text{ are real-valued continuous functions on } [-\pi, \pi] \right\}$$

(1) For $A(\theta)$, $B(\theta) \in V$, we define

$$(A, B) = \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{tr}(A(\theta)^t B(\theta))\, d\theta.$$

Show that this is an inner product in the real vector space V. Here, $A(\theta)^t$ is the transposed matrix of $A(\theta)$, and $\mathrm{tr}(A(\theta))$ is the trace of $A(\theta)$.

(2) For a nonnegative integer n, we define

$$O_n(\theta) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}.$$

Prove that $(O_m, O_n) = \delta_{mn}$ for any nonnegative integers m, n. Here, δ_{mn} denotes Kronecker's delta.

(3) We fix $A(\theta) \in V$. For $(n+1)$ real numbers $a_0, a_1, \ldots, a_n$, we define

$$Q(a_0, a_1, \ldots, a_n) = (A - \sum_{k=0}^{n} a_k O_k, A - \sum_{k=0}^{n} a_k O_k).$$

Prove that $Q(a_0, a_1, \ldots, a_n)$ takes its minimum when $a_k = (A, O_k)$ for $k = 0, 1, \ldots, n$.

$\boxed{9}$ Let (X, d) be a metric space and let $C \subset X$ be a non-empty closed set. We define a function $f : X \to \mathbb{R}$ by

$$f(x) = \inf\{d(x, y) \mid y \in C\} \quad (x \in X).$$

(1) Prove that f is continuous.

(2) Prove that $f(x) = 0$ if and only if $x \in C$.

$\boxed{10}$ Let K be a field, and let U be a finite-dimensional vector space over K. Let V_1, V_2, V_3 be subspaces of U.

(1) Prove that if a subspace V of U is included in the union $V_1 \cup V_2$, then $V \subset V_1$ or $V \subset V_2$.

(2) Assume that K is the real field. Prove that if a subspace V of U is included in the union $V_1 \cup V_2 \cup V_3$, then V is included in one of the three V_1, V_2, V_3.

(3) Does the statement "If a subspace V of U is included in $V_1 \cup V_2 \cup V_3$, then V is included in one of the three V_1, V_2, V_3" hold for any field K and for any finite-dimensional subspaces V_j $(j = 1, 2, 3)$? Give a proof if this statement is true, otherwise give a counter-example of K and V, V_1, V_2, V_3.