March, 2013

$\boxed{1}$ Let α be a nonzero real number, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonzero real numbers satisfying

$$\lim_{n\to\infty} \frac{\sin a_n}{a_n} = \alpha.$$

(1) Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.

(2) When $\alpha = 1$, show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges, and find the value of $\lim\limits_{n\to\infty} a_n$.

$\boxed{2}$

(1) For a positive constant c, define a function $f_c : (-\pi, \pi) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by

$$f_c(x) = \arctan(c \tan \tfrac{x}{2}).$$

Express its derivative $f_c'(x)$ in terms of $\cos x$.

(2) For a constant $\alpha > 1$, prove the formula

$$\int_0^{\pi} \frac{dx}{\alpha + \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}.$$

(3) For a constant $\alpha > 1$, find the value of the definite integral

$$\int_0^{\pi} \frac{dx}{(\alpha + \cos x)^2}.$$

$\boxed{3}$ Let $M_4(\mathbb{R})$ be the vector space of 4×4 real matrices.

(1) For real numbers a and b, find the determinant $|A|$ of the matrix

$$A = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix}.$$

(2) Is the subset

$$W = \left\{ A = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R},\ |A| = 0 \right\}$$

of $M_4(\mathbb{R})$ a linear subspace of $M_4(\mathbb{R})$? Give your answer with a reason.

(3) Prove that the following two 4×4 matrices are linearly independent over $\mathbb{R}$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(4) Prove that $|B| \geq 0$ for any matrix B in the linear subspace spanned by B_1 and B_2 given in (3).

$\boxed{4}$ Let $\mathbb{R}^n$ be the n-dimensional real vector space equipped with the standard inner product $\boldsymbol{u} \cdot \boldsymbol{v}$, and let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Set $\boldsymbol{f} = \sum_{i=1}^{n} \boldsymbol{e}_i$ and

$$W = \{\boldsymbol{v} \in \mathbb{R}^n \mid \boldsymbol{v} \cdot \boldsymbol{f} = 0\}.$$

For a nonzero vector $\boldsymbol{a} \in \mathbb{R}^n$, let $r_{\boldsymbol{a}}$ denote the map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$r_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{x} - 2\frac{\boldsymbol{a} \cdot \boldsymbol{x}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a}.$$

(1) Prove that W is a linear subspace of $\mathbb{R}^n$.

(2) Prove that $r_{\boldsymbol{a}}$ is linear.

(3) Prove that $\{\boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2 - \boldsymbol{e}_3, \ldots, \boldsymbol{e}_{n-1} - \boldsymbol{e}_n\}$ is a basis of W.

(4) For a nonzero vector $\boldsymbol{a} \in W$, prove that $r_{\boldsymbol{a}}(W) = W$.

(5) Assume that $n = 4$. Find the representation matrix of $r_{\boldsymbol{e}_2 - \boldsymbol{e}_3}$ with respect to the basis $\{\boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2 - \boldsymbol{e}_3, \boldsymbol{e}_3 - \boldsymbol{e}_4\}$ of W.

$\boxed{5}$ Let n be a positive integer, and let $F = \{0, 1\}$. Define a map $d : F^n \times F^n \to \mathbb{Z}$ by

$$d(\boldsymbol{u}, \boldsymbol{v}) = \Big|\{i \mid 1 \le i \le n,\ u_i \ne v_i\}\Big|,$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in F^n$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in F^n$, and $|S|$ denotes the cardinality of a finite set S. For an integer k with $0 \le k \le n$ and $\boldsymbol{u} \in F^n$, define

$$N_k(\boldsymbol{u}) = \{\boldsymbol{v} \in F^n \mid d(\boldsymbol{u}, \boldsymbol{v}) = k\}.$$

(1) Let k be an integer with $1 \le k \le n$ and let $\boldsymbol{u}, \boldsymbol{v} \in F^n$. Assume that $d(\boldsymbol{u}, \boldsymbol{v}) = k$. Find $|N_{k-1}(\boldsymbol{u}) \cap N_1(\boldsymbol{v})|$.

(2) Let i and j be integers with $0 \le i \le n$, $0 \le j \le n$ and let $\boldsymbol{u}, \boldsymbol{v} \in F^n$. Let $d(\boldsymbol{u}, \boldsymbol{v}) = k$. Find $|N_i(\boldsymbol{u}) \cap N_j(\boldsymbol{v})|$.

(3) Let i and j be integers with $0 \le i < j \le n$ and let $\boldsymbol{u}, \boldsymbol{v} \in F^n$. Prove that $d(\boldsymbol{x}, \boldsymbol{y}) < 2j$ for any $\boldsymbol{x}, \boldsymbol{y} \in N_i(\boldsymbol{u}) \cap N_j(\boldsymbol{v})$.

$\boxed{6}$ The covariance of two random variables X and Y is defined by

$$\mathbf{Cov}(X, Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])],$$

where $\mathbf{E}[Z]$ stands for the mean value of a random variable Z. Answer the following questions.

(1) Assume that two random variables X and Y are independent. Prove that $\mathbf{Cov}(X, Y) = 0$.

(2) Assume that random variables X and Y take values in $\{0, 1\}$. Prove that X and Y are independent if $\mathbf{Cov}(X, Y) = 0$.

(3) Assume that random variables X and Y take values in $\{-1, 0, 1\}$ and that $\mathbf{E}[X] = \mathbf{E}[Y] = \mathbf{Cov}(X, Y) = 0$. Are X and Y independent ? Give your answer with a reason.

$\boxed{7}$ Consider a solution $x = x(t)$ of the differential equation

$$\frac{d^2x}{dt^2} + \gamma(t)x = 0,$$

where $\gamma(t)$ is a real-valued continuous function on $\mathbb{R}$.

(1) When $\gamma(t)$ is a non-positive constant function, prove that if a solution $x(t)$ satisfies $x(0) = x(1) = 0$, then $x(t)$ identically equals 0.

(2) When $\gamma(t)$ is a positive constant function, find a solution $x(t)$ with $x(0) = x(1) = 0$ which does not identically equal 0.

(3) When $\gamma(t)$ is a positive function, prove that if a solution $x(t)$ satisfies $x(t+1) = 4x(t)$, then $x(t)$ vanishes at least at one point in $[0, 1]$.

$\boxed{8}$ Consider the the meromorphic function

$$f(z) = \frac{z}{(z^2+4)(z^2-2z+10)}$$

on the complex plane.

(1) Find all the poles of $f(z)$ and their residues.

(2) Let C_R be the semi-circle given by $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, for a positive number R. Prove that the complex line integral

$$I_R = \int_{C_R} f(z)\, dz$$

tends to 0 as $R \to +\infty$.

(3) Find the value of the definite integral

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+4)(x^2-2x+10)}\, dx.$$

$\boxed{9}$ Let a, b, c, d be real numbers with $a < b$ and $c < d$.

(1) Prove that the open interval (a, b) and $\mathbb{R}$ are homeomorphic.

(2) Prove that the closed interval $[a, b]$ and $\mathbb{R}$ are not homeomorphic.

(3) Prove that the open interval (a, b) and the interval $[c, d) = \{x \in \mathbb{R} \mid c \leq x < d\}$ are not homeomorphic.

$\boxed{10}$ Let G be a finite group, A be a normal subgroup of G, and z be an element of G of order 2 satisfying

$$C_G(z) \cap A = \{e\},$$

where e is the identity element of G and $C_G(z) = \{g \in G \mid gz = zg\}$.

(1) Prove that the cardinality $|A|$ of A is odd.

(2) Prove that $\{x^z x^{-1} \mid x \in A\} = A$, where $x^z = z^{-1}xz$.

(3) Prove that A is abelian.